10 SECTION

Discrete Subgroups of Iso (R^2)

♣ In Section 9, we obtained a classification of all finite subgroups of the group of isometries $Iso\,(\mathbf{R}^2)$ of $\mathbf{R}^2$ by studying symmetry groups of regular polygons. We saw that such subgroups cannot contain translations or glides, a fact that is intimately connected to boundedness of regular polygons. If we want to include translations and glides in our study, we have to start with unbounded plane figures and their symmetry groups. It turns out that classification of these subgroups is difficult unless we assume that the subgroup $G \subset Iso\,(\mathbf{R}^2)$ does not contain rotations of arbitrarily small angle and translations of arbitrarily small vector length. (As we will see later, we do not have to impose any condition on reflections and glides.) Groups $G \subset Iso\,(\mathbf{R}^2)$ satisfying this condition are called *discrete*. In this section we give a complete classification of discrete subgroups of $Iso\,(\mathbf{R}^2)$. Just as cyclic and dihedral groups can be viewed as orientation-preserving and full symmetry groups of regular polygons, we will visualize these groups as symmetries of frieze and wallpaper patterns. Thus, next time you look at a wallpaper pattern, you should be able to write down generators and relations for the corresponding symmetry group!

Let $G \subset Iso\,(\mathbf{R}^2)$ be a discrete group. Assume that G contains a translation $T_v \in G$ that is not the identity ($v \neq 0$). All powers of T_v are then contained in G (by the group property): $T_v^k \in G$, $k \in \mathbf{Z}$. Since $T_v^k = T_{kv}$, we see that all these are mutually distinct. It follows that G must be infinite. (The same conclusion holds for glides, since the square of a glide is a translation.) We see that the presence of translations or glides makes G infinite. The following question arises naturally: If we are able to excise the translations from G, is the remaining "part" of G finite? The significance of an affirmative answer is clear, since we just classified all finite subgroups of *Iso* ($\mathbf{R}^2$). This gives us a good reason to look at translations first.

Let $\mathcal{T}$ be the group of translations in $\mathbf{R}^2$. It is clearly a subgroup of *Iso* ($\mathbf{R}^2$). From now on we agree that for a translation $T_v \in \mathcal{T}$, we draw the translation vector v from the origin. Associating to T_v the vector v (just made unique) gives the map

$$\varphi : \mathcal{T} \to \mathbf{R}^2,$$

defined by

$$\varphi(T_v) = v, \quad v \in \mathbf{R}^2.$$

Since

$$T_{v_1} \circ T_{v_2} = T_{v_1+v_2}, \quad v_1, v_2 \in \mathbf{R}^2,$$

and

$$(T_v)^{-1} = T_{-v}, \quad v \in \mathbf{R}^2,$$

we see that φ is an isomorphism. Summarizing, the translations in *Iso* ($\mathbf{R}^2$) form a subgroup $\mathcal{T}$ that is isomorphic with the additive group $\mathbf{R}^2$.

Let $G \subset$ *Iso* ($\mathbf{R}^2$) be a discrete group. The translations in G form a subgroup $T = G \cap \mathcal{T}$ of G. Since G is discrete, so is T. The isomorphism $\varphi : \mathcal{T} \to \mathbf{R}^2$ maps T to a subgroup denoted by $L_G \subset \mathbf{R}^2$. This latter group is also discrete in the sense that it does not contain vectors of arbitrarily small length. By definition, L_G is the group of vectors $v \in \mathbf{R}^2$ such that the translation T_v is in G. We now classify the possible choices for L_G.

Theorem 5.

Let L be a discrete subgroup of $\mathbf{R}^2$. Then L is one of the following:

1. $L = \{0\}$;
2. *L consists of integer multiples of a nonzero vector $v \in \mathbf{R}^2$:*

$$L = \{kv \mid k \in \mathbf{Z}\};$$

3. *L consists of integral linear combinations of two linearly independent vectors $v,w \in \mathbf{R}^2$:*

$$L = \{kv + lw \mid k,l \in \mathbf{Z}\}.$$

Proof.
We may assume that L contains a nonzero vector $v \in \mathbf{R}^2$. Let $l = \mathbf{R} \cdot v$ be the line through v. Since L is discrete, there is a vector in $l \cap L$ of shortest length. Changing the notation if necessary, we may assume that this vector is v. Let w be any vector in $l \cap L$. We claim that w is an integral multiple of v. Indeed, $w = av$ for some $a \in \mathbf{R}$ since w is in l. Writing

$$a = k + r,$$

where k is an integer and $0 \leq r < 1$, we see that $w - kv = (a-k)v = rv$ is in L. On the other hand, if $r \neq 0$, then the length of rv is less than that of v, contradicting the minimality of v. Thus $r = 0$, and $w = kv$, an integer multiple of v. If there are no vectors in L outside of l, then we land in case 2 of the theorem.

Finally, assume that there exists a vector $w \in L$ not in l. The vectors v and w are linearly independent, so that they span a parallelogram P. Since P is bounded, it contains only finitely many elements of L. Among these, there is one whose distance to the line l is positive, but the smallest possible. By changing w (and P), we may assume that this vector is w. We claim now that there are no vectors of L in P except for its vertices. ¬ Assume the contrary and let $z \in L$ be a vector in P. Due to the minimal choice of v and w, this is possible only if z terminates at a point on the opposite side of v or w. In the first case, $z - w \in L$ would be a vector shorter than v; in the second, z would be closer to l than w. ¬ Summarizing, we conclude that there are two linearly independent vectors v and w that span a parallelogram P such that P contains no vectors in L except for its vertices. Clearly, $\{kv + lw \mid k, l \in \mathbf{Z}\}$ is contained in L. To land in case 3 we now claim that every vector z in L is an integral linear combination of v and w. By linear independence, z is certainly a linear combination

$$z = av + bw$$

of v and w with *real* coefficients $a, b \in \mathbf{R}$. We now write

$$a = k + r \quad \text{and} \quad b = l + s,$$

where $k, l \in \mathbf{Z}$ and $0 \leq r, s < 1$. The vector $z - kv - lw = rv + sw$ is in L and is contained in P. The only way this is possible is if $r = s = 0$ holds. Thus $z = kv + lw$, and we are done. ■

We now return to our discrete group $G \subset Iso\,(\mathbf{R}^2)$ and see that we have three choices for L_G. If $L_G = \{0\}$, then G does not contain any translations (or glides, since the square of a glide is a translation). In this case, G consists of rotations and reflections only. By a result of the previous section, the rotations in G have the same center, say, p_0. Since G is discrete, it follows that G contains only finitely many rotations. If $R_l \in G$ is a reflection, then l must go through p_0, since otherwise, $R_l(p_0)$ would be the center of another rotation in G. Finally, since the composition of two reflections in G is a rotation in G, there may be only at most as many reflections in G as rotations (cf. the proof of Theorem 4). Summarizing, we obtain that if G is a discrete group of isometries with $L_G = \{0\}$ then G must be finite. In the second case T, the group of all translations in G, is generated by T_v, and we begin to suspect that G is the symmetry group of a *frieze pattern*. Finally, in the third case T is generated by T_v and T_w, and T is best viewed by its φ-image $L_G = \{kv + lw \mid k, l \in \mathbf{Z}\}$ in $\mathbf{R}^2$. We say that L_G is a *lattice* in $\mathbf{R}^2$ and G is a (2-dimensional) *crystallographic group*. Since any wallpaper pattern repeats itself in two different directions, we see that their symmetry groups are crystallographic.

We now turn to the process of "excising" the translation part from G. To do this, we need some preparations. Recall that at the discussion of translations we agreed to draw the vectors v from the origin so that the translation T_v by the vector $v \in \mathbf{R}^2$ acts on $p \in \mathbf{R}^2$ by $T_v(p) = p + v$. Now given any *linear* transformation $A : \mathbf{R}^2 \to \mathbf{R}^2$ (that is, $A(v_1 + v_2) = A(v_1) + A(v_2)$, $v_1, v_2 \in \mathbf{R}^2$, and $A(rv) = rA(v)$, $r \in \mathbf{R}$, $v \in \mathbf{R}^2$), we have the commutation rule

$A \circ T_v = T_{A(v)} \circ A$. Indeed, evaluating the two sides at $p \in \mathbf{R}^2$, we get

$$(A \circ T_v)(p) = A(T_v(p)) = A(p + v) = A(p) + A(v)$$

and

$$(T_{A(v)} \circ A)(p) = T_{A(v)}(A(p)) = A(p) + A(v).$$

Let $O(\mathbf{R}^2)$ denote the group of isometries in $Iso\,(\mathbf{R}^2)$ that leave the origin fixed. $O(\mathbf{R}^2)$ is called the *orthogonal group*. From the classification of the plane isometries, it follows that the elements of $O(\mathbf{R}^2)$ are linear.

Remark.
♠ We saw above that a direct isometry in $O(\mathbf{R}^2)$ is a rotation R_θ. These rotations form the *special orthogonal group* $SO(\mathbf{R}^2)$, a subgroup of $O(\mathbf{R}^2)$. Associating to R_θ the complex number $z(\theta)$ establishes an isomorphism between $SO(\mathbf{R}^2)$ and S^1. Any opposite isometry in $O(\mathbf{R}^2)$ can be written as a rotation followed by conjugation. Thus *topologically* $O(\mathbf{R}^2)$ is the disjoint union of two circles.

♣ Occasionally, it is convenient to introduce superscripts $\pm$ to indicate whether the isometries are direct or opposite. Thus $Iso^+(\mathbf{R}^2)$ denotes the set of direct isometries in $Iso\,(\mathbf{R}^2)$. Note that it is a subgroup, since the composition and inverse of direct isometries are direct. $Iso^-(\mathbf{R}^2)$ is not a subgroup but a topological copy of $Iso^+(\mathbf{R}^2)$.

♡ The elements of $O(\mathbf{R}^2)$ are linear, so that the commutation rule above applies. We now define a homomorphism

$$\psi : Iso\,(\mathbf{R}^2) \to O(\mathbf{R}^2)$$

as follows: Let $S \in Iso\,(\mathbf{R}^2)$ and denote by v the vector that terminates at $S(0)$. The composition $(T_v)^{-1} \circ S$ fixes the origin so that it is an element of $O(\mathbf{R}^2)$. We define $\psi(S) = (T_v)^{-1} \circ S$. To prove that ψ is a homomorphism, we first write $(T_v)^{-1} \circ S = U \in O(\mathbf{R}^2)$, so that $S = T_v \circ U$. This decomposition is unique in the sense that if $S = T_{v'} \circ U'$ with $v' \in \mathbf{R}^2$ and $U' \in O(\mathbf{R}^2)$, then $v = v'$ and $U = U'$. Indeed, $T_v \circ U = T_{v'} \circ U'$ implies that $(T_{v'})^{-1} \circ T_v = U' \circ U^{-1}$. The right-hand side fixes the origin so that the left-hand side, which is a translation, must be the identity. Uniqueness follows.

Using the notation we just introduced, we have $\psi(S) = U$, where $S = T_v \circ U$. Now let $S_1 = T_{v_1} \circ U_1$ and $S_2 = T_{v_2} \circ U_2$, where $v_1, v_2 \in \mathbf{R}^2$ and $U_1, U_2 \in O(\mathbf{R}^2)$. For the homomorphism property, we need to show that $\psi(S_2 \circ S_1) = \psi(S_2) \circ \psi(S_1)$. By definition, $\psi(S_1) = U_1$ and $\psi(S_2) = U_2$, so that the right-hand side is $U_2 \circ U_1$. As for the left-hand side, we first look at the composition

$$S_2 \circ S_1 = T_{v_2} \circ U_2 \circ T_{v_1} \circ U_1.$$

Using the commutation rule for the linear U_2, we have $U_2 \circ T_{v_1} = T_{U_2(v_1)} \circ U_2$. Inserting this, we get

$$S_2 \circ S_1 = T_{v_2} \circ T_{U_2(v_1)} \circ U_2 \circ U_1.$$

Taking ψ of both sides amounts to deleting the translation part:

$$\psi(S_2 \circ S_1) = U_2 \circ U_1.$$

Thus ψ is a homomorphism.

ψ is onto since it is identity on $O(\mathbf{R}^2) \subset \mathit{Iso}\,(\mathbf{R}^2)$. The kernel of ψ consists of translations:

$$\ker \psi = \mathcal{T}.$$

In particular, $\mathcal{T} \subset \mathit{Iso}\,(\mathbf{R}^2)$ is a normal subgroup. Having constructed $\psi : \mathit{Iso}\,(\mathbf{R}^2) \to O(\mathbf{R}^2)$, we return to our discrete group $G \subset \mathit{Iso}\,(\mathbf{R}^2)$. The ψ-image of G is called the *point-group* of G, denoted by $\bar{G} = \psi(G)$. The kernel of $\psi|G$ is all translations in G, that is, T. Thus, we have the following:

$$\psi|G : G \to \bar{G} \subset O(\mathbf{R}^2)$$

and

$$ker(\psi|G) = T.$$

For nontrivial L_G, the point-group $\bar{G}$ interacts with L_G in a beautiful way:

Theorem 6.
$\bar{G}$ leaves L_G invariant.

Proof.
Let $U \in \bar{G}$ and $v \in L_G$. We must show that $U(v) \in L_G$. Since $U \in \bar{G}$, there exists $S \in G$, with $S = T_w \circ U$ for some $w \in \mathbf{R}^2$. The assumption $v \in L_G$, means $T_v \in G$, and what we want to conclude, $U(v) \in L_G$, means $T_{U(v)} \in G$. We compute

$$\begin{aligned} T_{U(v)} &= T_{U(v)} \circ T_w \circ (T_w)^{-1} \\ &= T_w \circ T_{U(v)} \circ (T_w)^{-1} \\ &= T_w \circ U \circ T_v \circ U^{-1} \circ (T_w)^{-1} \\ &= S \circ T_v \circ S^{-1} \in G, \end{aligned}$$

where the last but one equality is because of the commutation relation

$$U \circ T_v = T_{U(v)} \circ U$$

as established above. The theorem follows. ■

$\bar{G}$ is discrete in the sense that is does not contain rotations with arbitrarily small angle. This follows from Theorem 6 if L_G is nontrivial. If L_G is trivial, then by a result of the previous section, G is finite, and so is its (isomorphic) image $\bar{G}$ under ψ. Since $\bar{G}$ is discrete and fixes the origin, it must be finite! Indeed, by now this argument should be standard. Let $R_\theta \in \bar{G}$ with θ being the smallest positive angle. Then any rotation in $\bar{G}$ is a multiple of R_θ. Moreover, using the division algorithm, we have $2\pi = n\theta + r$, $0 \leq r < \theta$, $n \in \mathbf{Z}$, and r must reduce to zero because of minimality of θ. Thus $\theta = 2\pi/n$, and the rotations form a cyclic group of order n. Finally, there cannot be infinitely many reflections, since otherwise their axes could get arbitrarily close to each other, and composing any two could give rotations of arbitrarily small angle. We thus accomplished our aim. $\bar{G}$ gives a finite subgroup in $O(\mathbf{R}^2)$ consisting of rotations and reflections only. In particular, if $L_G = \{0\}$—that is, if G contains no nontrivial translations—then the kernel of $\psi|G$ is trivial and so $\psi|G$ maps G isomorphically onto $\bar{G}$. In particular, G is finite. By Theorem 4 of Section 9, G is cyclic or dihedral.

We are now ready to classify the possible frieze patterns, of which there are seven. According to Theorem 6, a frieze group G keeps the line c through L_G invariant, and the group of translations T in G is an infinite cyclic subgroup generated by a shortest translation, say, τ, in the direction of c. The line c is called the "center" of the frieze group. In addition to T, the only nontrivial direct isometries are rotations with angle π, called "half-turns," and their center must be on c. The only possible opposite isometries are reflection to c, reflections to lines perpendicular to c, and glides along c. In the classification below we use the following notations: If G contains a half-turn, we denote its center by $p \in c$. If G does not contain any half-turns, but contains reflections to lines perpendicular to c, the axis of reflection is denoted by l, and p is the intersection point of l and c. Otherwise p is any point on c. Let $p_n = \tau^n(p)$, $n \in \mathbf{Z}$, and $m =$ the midpoint of the segment connecting p_0 and p_1. Finally, let $m_n = \tau^n(m)$, the midpoint of the segment connecting p_n and p_{n+1} (Figure 10.1).

We are now ready to start. First we classify the frieze groups that contain only direct isometries:

1. $G = T = \langle \tau \rangle$, so that G contains[1] no half-turns, reflections or glide reflections.
2. $G = \langle \tau, H_p \rangle$. Aside from translations, G contains the half-turns $\tau^n \circ H_p$. For $n = 2k$ even, $\tau^{2k} \circ H_p$ has center at p_k, and for $n = 2k+1$ odd, $\tau^{2k+1} \circ H_p$ has center at m_k.

It is not hard to see that these are all the frieze groups that contain only direct isometries. We now allow the presence of opposite isometries.

3. $G = \langle \tau, R_c \rangle$. Since $R_c^2 = I$ and $\tau \circ R_c = R_c \circ \tau$, aside from T, this group consists of glides $\tau^n \circ R_c$ mapping p to p_n.

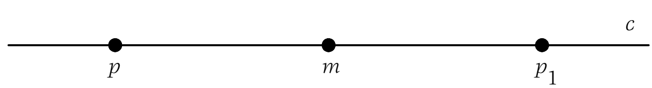

Figure 10.1

[1] If $\Gamma \subset Iso\,(\mathbf{R}^2)$, then $\langle \Gamma \rangle$ denotes the smallest subgroup in $Iso\,(\mathbf{R}^2)$ that contains Γ. We say that Γ generates $\langle \Gamma \rangle$ (cf. "Groups" in Appendix B).

4. $G = \langle \tau, R_l \rangle$. Since $R_l^2 = I$ and $R_l \circ \tau = \tau^{-1} \circ R_l$, aside from T, G consists of reflections $\tau^n \circ R_l$. The axes are perpendicular to c, and according to whether $n = 2k$ (even) or $n = 2k + 1$ (odd), the intersections are p_k or m_k.
5. $G = \langle \tau, H_p, R_c \rangle$. We have $H_p \circ R_c = R_l \circ R_c \circ R_c = R_l \in G$. In addition to this and τ, G includes the glides $\tau^n \circ R_c$ (sending p to p_n) and $\tau^n \circ R_l$ discussed above.
6. $G = \langle \tau, H_p, R_{l'} \rangle$. Then l' must intersect c perpendicularly at the midpoint of p and m_k, for some $k \in \mathbf{Z}$.
7. $G = \langle G_{c,v} \rangle$ is generated by the glide $G_{c,v}$ with $G_{c,v}^2 = \tau$.

Figure 10.2 depicts the seven frieze patterns. (The pictures were produced with Kali (see Web Site 2), written by Nina Amenta of the Geometry Center at the University of Minnesota.) Which corresponds to which in the list above?

The fact that the point-group $\bar{G}$ leaves L_G invariant imposes a severe restriction on G if L_G is a lattice, the case we turn to next.

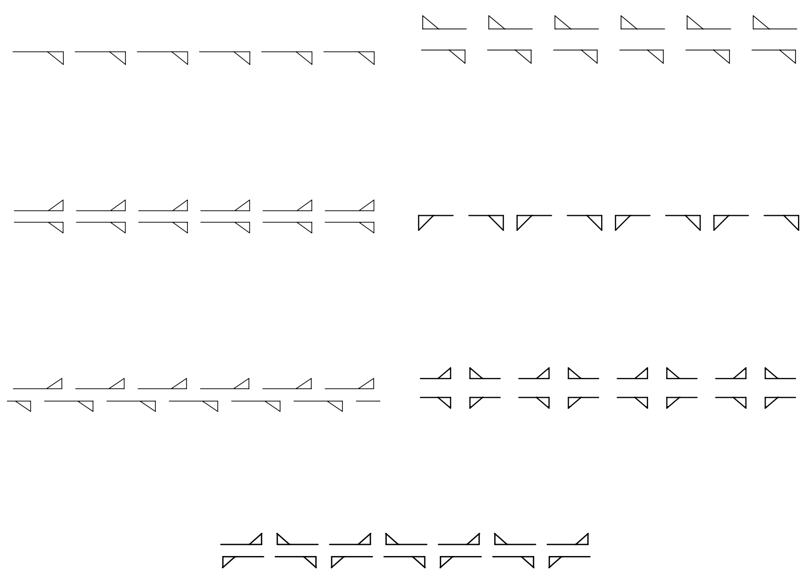

Figure 10.2

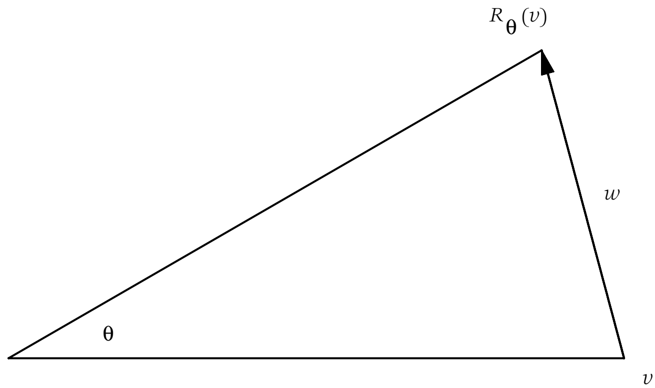

Figure 10.3

Crystallographic Restriction.

Assume that G *is crystallographic. Let* $\bar{G}$ *denote its point-group. Then every rotation in* $\bar{G}$ *has order* 1,2,3,4, *or* 6, *and* $\bar{G}$ *is* C_n *or* D_n *for some* $n = 1,2,3,4,$ *or* 6.

Proof.

As usual, let R_θ be the smallest positive angle rotation in $\bar{G}$, and let v be the smallest length nonzero vector in L_G. Since L_G is $\bar{G}$-invariant, $R_\theta(v) \in L_G$. Consider $w = R_\theta(v) - v \in L_G$ (Figure 10.3).

Since v has minimal length, $|v| \le |w|$. Thus,

$$\theta \ge 2\pi/6,$$

and so R_θ has order ≤ 6. The case $\theta = 2\pi/5$ is ruled out since $R_\theta^2(v) + v$ is shorter than v (Figure 10.4).

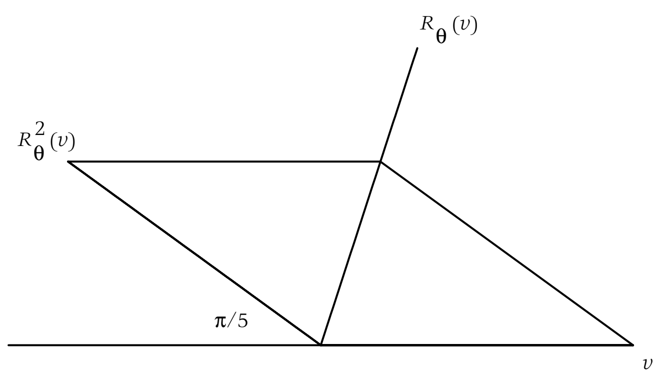

Figure 10.4

The first statement follows. The second follows from the classification of finite subgroups of $Iso(\mathbf{R}^2)$ in the previous section. ■

Remark.
For an algebraic proof of the crystallographic restriction, consider the trace $\mathrm{tr}\,(R_\theta)$ of $R_\theta \in \bar{G}$, $0 < \theta \leq \pi$. With respect to a basis in L_G, the matrix of R_θ has integral entries (Theorem 6). Thus, $\mathrm{tr}\,(R_\theta)$ is an integer. On the other hand, with respect to an orthonormal basis, the matrix of R_θ has diagonal entries both equal to $\cos(\theta)$. In particular, $\mathrm{tr}\,(R_\theta) = 2\cos(\theta)$. Thus, $2\cos(\theta)$ is an integer, and this is possible only for $n = 2, 3, 4$, or 6.

Example
If $\omega = z(2\pi/n)$ is a primitive nth root of unity, then $\mathbf{Z}[\omega]$ is a lattice iff $n = 3, 4$, or 6. Indeed, the rotation $R_{2\pi/n}$ leaves $\mathbf{Z}[\omega]$ invariant, since it is multiplication by ω. By the crystallographic restiction, $n = 3, 4$, or 6. How do the tesselations look for $n = 3$ and $n = 6$? □

The absence of order-5 symmetries in a lattice must have puzzled some ancient ornament designers. We quote here from Hermann Weyl's *Symmetry*: "The Arabs fumbled around much with the number 5, but they were of course never able honestly to insert a central symmetry of 5 in their ornamental designs of double infinite rapport. They tried various deceptive compromises, however. One might say that they proved experimentally the impossibility of a pentagon in an ornament."

Armed with the crystallographic restriction, we now have the tedious task of considering all possible scenarios for the point-group $\bar{G}$ and its relation to L_G. This was done in the nineteenth century by Fedorov and rediscovered by Polya and Niggli in 1924. A description of the seventeen crystallographic groups that arise are listed as follows:

Generators for the 17 Crystallographic Groups

1. Two translations.
2. Three half-turns.

3. Two reflections and a translation.
4. Two parallel glides.
5. A reflection and a parallel glide.
6. Reflections to the four sides of the rectangle.
7. A reflection and two half-turns.
8. Two perpendicular glides.
9. Two perpendicular reflections and a half-turn.
10. A half-turn and a quarter-turn.
11. Reflections in the three sides of a $(\pi/4, \pi/4, \pi/2)$ triangle.
12. A reflection and a quarter-turn.
13. Two rotations through $2\pi/3$.
14. A reflection and a rotation through $2\pi/3$.
15. Reflections in the three sides of an equilateral triangle.
16. A half-turn and a rotation through $2\pi/3$.
17. Reflections is in the three sides of a $(\pi/6, \pi/3, \pi/2)$ triangle.

Remark.
The following construction sheds some additional light on the geometry of crystallographic groups. Let G be crystallographic and assume that G contains rotations other than half-turns. Let $R_{2\alpha}(p) \in G$, $0 < \alpha < \pi/2$, be a rotation with integral π/α (cf. the proof of Theorem 4 of Section 9). Let $R_{2\beta}(q) \in G$, $0 < \beta < \pi/2$, be another rotation with integral π/β such that $d(p, q)$ is minimal. ($R_{2\beta}(q)$ exists since G is crystallographic.) Let l denote the line through p and q. Write $R_{2\alpha}(p) = R_{l'} \circ R_l$, where l' meets l at p and the angle from l to l' is α. Similarly, $R_{2\beta}(q) = R_l \circ R_{l''}$, where l'' meets l at q and the angle from l'' to l is β. Since $\alpha + \beta < \pi$, the lines l' and l'' intersect at a point, say, r. In fact, r is the center of the rotation $R_{2\gamma}(r) = (R_{2\alpha}(p) \circ R_{2\beta}(q))^{-1} = R_{l''} \circ R_{l'}$. Since α, β, and γ are the interior angles of the triangle $\triangle pqr$, we have $\alpha + \beta + \gamma = \pi$. On the other hand, since G is discrete, π/γ is rational. It is easy to see that minimality of $d(p, q)$ implies that π/γ is integral. We obtain that

$$\frac{\alpha}{\pi} + \frac{\beta}{\pi} + \frac{\gamma}{\pi} = 1,$$

where the terms on the left-hand side are reciprocals of integers. Since $\pi/\alpha, \pi/\beta \geq 3$ (and $\pi/\gamma \geq 2$), the only possibilities are $\alpha =$

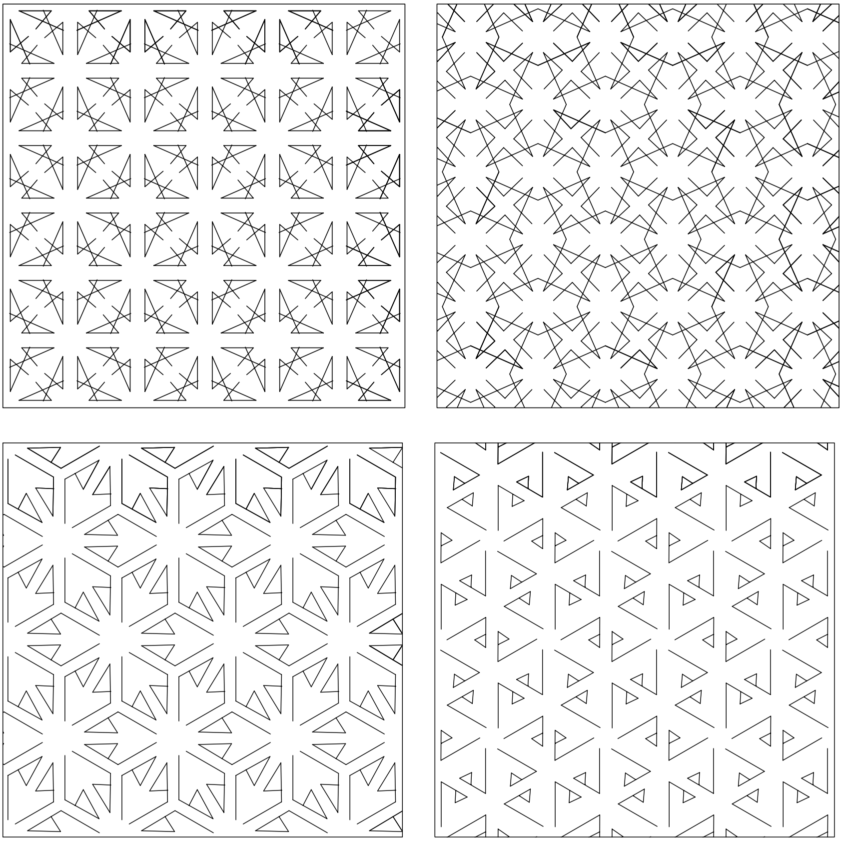

Figure 10.5

$\beta = \gamma = \pi/3$; $\alpha = \beta = \pi/4$, $\gamma = \pi/2$; and $\alpha = \pi/6$, $\beta = \pi/3$, $\gamma = \pi/2$. (Which corresponds to which in the list above?)

As noted above, these groups can be visualized by patterns covering the plane with symmetries prescribed by the acting crystallographic group. Figure 10.5 shows a sample of four patterns (produced with Kali).

Symmetric patterns[2] date back to ancient times. They appear in virtually all cultures; on Greek vases, Roman mosaics, in the thirteenth century Alhambra at Granada, Spain, and on many other Muslim buildings.

[2]For a comprehensive introduction see B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, Freeman, 1987.

To get a better view of the repetition patterns, we introduce the concept of fundamental domain. First, given a discrete group $G \subset$ *Iso* ($\mathbf{R}^2$), a *fundamental set* for G is a subset F of $\mathbf{R}^2$ which contains exactly one point from each orbit

$$G(p) = \{S(p) \mid S \in G\}, \quad p \in \mathbf{R}^2.$$

A *fundamental domain* F_0 for G is a domain (that is, a connected open set) such that there is a fundamental set F between F_0 and its closure[3] $\bar{F}_0$; that is, $F_0 \subset F \subset \bar{F}_0$, and the 2-dimensional area of the boundary $\partial F_0 = \bar{F}_0 - F_0$ is zero.

The simplest example of a fundamental set (domain) is given by the translation group $G = T = \langle T_v, T_w \rangle$. In this case, a fundamental domain F_0 is the open parallelogram spanned by v and w. A fundamental set F is obtained from F_0 by adding the points tv and tw, $0 \leq t < 1$. By the defining property of the fundamental set, the "translates" $S(F)$, $S \in G$, tile[4] or, more sophisticatedly, *tessellate* $\mathbf{R}^2$. (Numerous tessellations appear in Kepler's *Harmonice Mundi*, which appeared in 1619.) If a pattern is inserted in F, translating it with G gives the wallpaper patterns that you see. You are now invited to look for fundamental sets in Figure 10.5!

Problems

1. Prove directly that any plane isometry that fixes the origin is linear.
2. Identify the frieze group that corresponds to the pattern in Figure 10.6.
3. Let $L \subset \mathbf{R}^2$ be a lattice. Show that half-turn around the midpoint of any two points of L is a symmetry of L.
4. Identify the discrete group G generated by the three half-turns around the midpoints of the sides of a triangle.

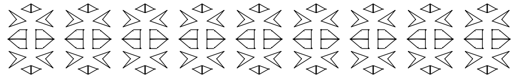

Figure 10.6

[3] See "Topology" in Appendix C.

[4] We assume that the tiles can be turned over; i.e., they are decorated on both sides.

Web Sites

1. www.geom.umn.edu/docs/doyle/mpls/handouts/node30.html
2. www.geom.umn.edu/apps/kali/start.html
3. www.math.toronto.edu/~coxeter/art-math.html
4. www.texas.net/escher/gallery
5. www.suu.edu/WebPages/MuseumGaller/Art101/aj-webpg.htm
6. www.geom.umn.edu/apps/quasitiler/start.html