To help the reader to feel the style of Abel's era and, at the same time, become acquainted with the modern interpretation of the subject, we will reproduce both proofs: the one that Eisenstein obtained one and a half centuries ago and the one recently found by Rosen.

Abel only proved the possibility of the division of the lemniscate into n equal parts with a ruler and compass for the indicated values of n. He did not prove that for the other values of n this is impossible. In [**C14**] it is shown that for the other values of n it is impossible to construct the coordinates of the points that divide the lemniscate into n equal parts with a ruler and compass. This, however, does not mean that for the other values of n it is impossible to divide the lemniscate into n equal parts with a ruler and compass if the lemniscate *is already drawn*. Indeed, use of the lemniscate *itself* provides us with additional possibilities for constructions. Considering the points of intersection of the straight lines and circles with the lemniscate one can, in general, construct more than just quadratic irrationalities.

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Before we plunge into the study of the equation for the division of the lemniscate, let us consider a simpler equation for the division of the circle. First, we will show how to solve in square roots the equation $x^{17} - 1 = 0$ by a quite elementary method, though this solution cannot be generalized to the equation for the division of the lemniscate. Next, we will discuss the approach to the study of the solvability of the equation $x^n - 1 = 0$ in square roots that can be generalized to the equation for the division of the lemniscate.

§4.1. Construction of a regular 17-gon. An elementary approach

The roots of the equation $x^n - 1 = 0$ are the vertices of a regular n-gon. Indeed, if $\varepsilon = \exp(2\pi i/n)$, then $\varepsilon, \varepsilon^2, \ldots, \varepsilon^n = 1$ are the roots of this equation. Dividing the polynomial $x^n - 1$ by $x - 1$ we get the polynomial $x^{n-1} + x^{n-2} + \cdots + x + 1$. Thus, if the equation

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = 0 \tag{1.1}$$

is solvable in square roots, then it is possible to construct a regular n-gon with a ruler and compass.

For $n = 3$ there is no problem, since the quadratic $x^2 + x + 1 = 0$ is, without doubt, solvable in square roots. For $n = 5$ equation (1.1) is also easy to solve. Indeed, the substitution $u = x + x^{-1}$ turns it into $u^2 + u - 1 = 0$.

For $n = 17$ it is not that easy to solve equation (1.1) in square roots. To do so, Gauss used a special partition of the numbers ε, ε^2, ε^3, ..., ε^{16} into groups, where $\varepsilon = \exp(2\pi i/17)$. To get such a partition, we enumerate the given numbers so that for a fixed l the root ε_{k+l} is obtained from ε_k in the same fashion, namely, by raising to a fixed power: $\varepsilon_{k+l} = (\varepsilon_k)^c$:

$$\varepsilon_k \varepsilon_l = \varepsilon_{k+l}.$$

Such a numeration can be obtained by setting $\varepsilon_k = \varepsilon^{g^k}$, where the residues of the numbers $1, g, g^2, \ldots, g^{15}$ after the division by 17 take all values from 1 to 16. It is

easy to see that $g = 3$ possesses this property. For $g = 3$ the numbers $\varepsilon_0, \ldots, \varepsilon_{15}$ and their respective values are written one under another in the following table:

ε	ε^3	ε^9	ε^{10}	ε^{13}	ε^5	ε^{15}	ε^{11}	ε^{16}	ε^{14}	ε^8	ε^7	ε^4	ε^{12}	ε^2	ε^6
ε_0	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8	ε_9	ε_{10}	ε_{11}	ε_{12}	ε_{13}	ε_{14}	ε_{15}

Let x_1 be the sum of the numbers ε_k with even indices k, and x_2 the sum of the numbers ε_k with odd indices k, i.e.,

$$x_1 = \varepsilon + \varepsilon^9 + \varepsilon^{13} + \varepsilon^{15} + \varepsilon^{16} + \varepsilon^8 + \varepsilon^4 + \varepsilon^2,$$
$$x_2 = \varepsilon^3 + \varepsilon^{10} + \varepsilon^5 + \varepsilon^{11} + \varepsilon^{14} + \varepsilon^7 + \varepsilon^{12} + \varepsilon^6.$$

The sum of all the roots of the equation $x^{17} - 1 = 0$ (the root $x = 1$ included) is equal to zero, hence, $x_1 + x_2 = -1$. Simple calculations show that $x_1 x_2 = -4$. Indeed, let $\alpha = 2\pi/17$. Then $\varepsilon^k = \cos k\alpha + i \sin k\alpha$; hence,

$$\varepsilon + \varepsilon^{16} = 2\cos\alpha, \qquad \varepsilon^9 + \varepsilon^8 = 2\cos 8\alpha,$$
$$\varepsilon^{13} + \varepsilon^4 = 2\cos 4\alpha, \quad \varepsilon^{15} + \varepsilon^2 = 2\cos 2\alpha,$$

i.e.,

$$x_1 = 2(\cos\alpha + \cos 8\alpha + \cos 4\alpha + \cos 2\alpha).$$

Similarly,

$$x_2 = 2(\cos 3\alpha + \cos 7\alpha + \cos 5\alpha + \cos 6\alpha).$$

Using the formula

$$2\cos p\alpha \cos q\alpha = \cos(p+q)\alpha + \cos(p-q)\alpha$$

we get

$$x_1 x_2 = 8(\cos\alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos 8\alpha) = 4(x_1 + x_2) = -4.$$

Thus, we can find x_1 and x_2 from the quadratic equation

$$x^2 + x - 4 = 0. \tag{1.2}$$

Since

$$\cos\alpha + \cos 2\alpha > 2\cos\frac{\pi}{4} = \sqrt{2} > -\cos 8\alpha$$

and $\cos 4\alpha > 0$, it follows that $x_1 > 0$. Hence, $x_2 = -\frac{4}{x_1} < 0$, i.e., x_1 is the positive root of equation (1.2) and x_2 is the negative root.

Denoting by y_1, y_3, y_2 and y_4 the sums of the numbers ε_k with indices whose residues modulo 4 are equal to 0, 1, 2 and 3, respectively, we get

$$y_1 = \varepsilon + \varepsilon^{13} + \varepsilon^{16} + \varepsilon^4 = 2(\cos\alpha + \cos 4\alpha),$$
$$y_2 = \varepsilon^9 + \varepsilon^{15} + \varepsilon^8 + \varepsilon^2 = 2(\cos 8\alpha + \cos 2\alpha),$$
$$y_3 = \varepsilon^3 + \varepsilon^5 + \varepsilon^{14} + \varepsilon^{12} = 2(\cos 3\alpha + \cos 5\alpha),$$
$$y_4 = \varepsilon^{10} + \varepsilon^{11} + \varepsilon^7 + \varepsilon^6 = 2(\cos 7\alpha + \cos 6\alpha).$$

It is clear that $y_1 + y_2 = x_1$ and $y_1 > y_2$, because $\cos\alpha > \cos 2\alpha$ and $\cos 4\alpha > \cos 8\alpha$. Moreover,

$$y_1 y_2 = 4(\cos\alpha + \cos 4\alpha)(\cos 8\alpha + \cos 2\alpha) = 2(\cos\alpha + \cdots + \cos 8\alpha) = -1.$$

Therefore, y_1 and y_2 satisfy the equation $y^2 - x_1 y - 1 = 0$. It is easy to verify that y_3 and y_4 satisfy the equation $y^2 - x_2 y - 1 = 0$; moreover, $y_3 > y_4$.

Finally, let us consider $z_1 = \varepsilon + \varepsilon^{16} = 2\cos\alpha$ and $z_2 = \varepsilon^{13} + \varepsilon^4 = 2\cos 4\alpha$, i.e., the sums of numbers ε_k with indices whose residues after the division by 8 are equal to 0 and 4, respectively. Then $z_1 > z_2$, $z_1 + z_2 = y_1$ and

$$z_1 z_2 = 4\cos\alpha\cos 4\alpha = 2(\cos 5\alpha + \cos 3\alpha) = y_3.$$

Therefore, z_1 is the largest root of the equation $z^2 - y_1 z + y_3 = 0$. Thus, the segment of length $z_1 = 2\cos(2\pi/17)$ can be constructed with a ruler and compass. Now it is clear how to construct a regular 17-gon.

§4.2. Construction of regular polygons. Elements of Galois theory

In the preceding section we showed how to solve in square roots the equation $x^{17} - 1 = 0$. Now we prove that for all numbers n of the form $2^n p_1 \cdots p_k$, where the p_i are distinct Fermat primes, the equation $x^n - 1 = 0$ is also solvable in square roots. Our exposition will be such that a good deal of it can be generalized to the case of the lemniscate almost without changes.

Assigning to every real number t the point with coordinates $(\cos t, \sin t)$, we get a parameterization of the unit circle C by real numbers. As a result, C turns into an abelian group with unit element $(1, 0)$.

Since

$$\cos(t+s) = \cos t\cos s - \sin t\sin s \quad \text{and} \quad \sin(t+s) = \sin t\cos s + \cos t\sin s,$$

the law of addition of points on this circle can be expressed as follows:

$$(a, b) + (c, d) = (ac - bd, ad + bc) = (f(a, b, c, d), g(a, b, c, d)).$$

It is easy to verify that

$$2(x, y) = (x, y) + (x, y) = (x^2 - y^2, 2xy)$$

and

$$3(x, y) = (x^3 - 3xy^2, 3x^2 y - y^3).$$

Similarly,

$$n(x, y) = (f_n(x, y), g_n(x, y)),$$

where f_n and g_n are polynomials with integer coefficients. From the relation $\cos n\varphi + i\sin n\varphi = (\cos\varphi + i\sin\varphi)^n$ we get

$$(2.1) \qquad f_n(x, y) = \frac{(x+iy)^n + (x-iy)^n}{2}, \quad g_n(x, y) = \frac{(x+iy)^n - (x-iy)^n}{2i}.$$

Let C_n be the set of points $(x, y) \in C$ such that $n(x, y) = (1, 0)$, i.e., $f_n(x, y) = 1$ and $g_n(x, y) = 0$. These points can serve as vertices of a regular n-gon. It is also clear that C_n is a subgroup of C isomorphic to $\mathbb{Z}/n\mathbb{Z}$, the additive group of residues modulo n.

Over $\mathbb{C}$, in addition to the points of C_n there are other solutions of the system

$$f_n(x, y) = 1, \quad g_n(x, y) = 0.$$

Let us find all these solutions. Using formulas (2.1) we can pass to an equivalent system of equations

$$(x+iy)^n = 1, \quad (x-iy)^n = 1.$$