and radii of α_z and α_w, respectively. By the theorem proved at the beginning of this section, show that the analytic continuation of $f(z)$ beyond α_z is given by

$$[f(z) - b]\left[\overline{f\left(a + \frac{r^2}{\bar{z} - \bar{a}}\right)} - \bar{b}\right] = R^2,$$

and explain why this relation is equivalent to the symmetry principle. *Hint:* Observe that a point z on α_z satisfies $r^2 = |z - a|^2 = (z - a)(\bar{z} - \bar{a})$ and that two points z_1 and z_2 which are inverse with respect to α_z are connected by the relation

$$(z_1 - a)(\bar{z}_2 - \bar{a}) = r^2.$$

2. Show that an analytic function which maps the circle $|z| < 1$ onto the n-times covered circle $|w| < 1$ must be a rational function with n poles.

3. Show that an analytic function $w = f(z)$ which maps the unit circle $|z| < 1$ onto the full w-plane from which the ray $-\infty \le w \le -\frac{1}{4}$ has been removed can be continued analytically beyond $|z| = 1$ by the relation

$$f(z) = \overline{f\left(\frac{1}{\bar{z}}\right)}.$$

Show further that $w = f(z)$ maps the full z-plane onto the doubly covered full w-plane and that, therefore, $f(z)$ must be a rational function with two poles. Verify that, with the additional conditions $f(0) = 0, f'(0) > 0, f(z)$ is of the form

$$f(z) = \frac{z}{(1 - z)^2}.$$

4. If $w = f(z)$ maps the ring $0 < \rho < |z| < 1$ onto the circle $|w| < 1$ from which the linear segment $-\alpha \le w \le \alpha$ $(0 < \alpha < 1)$ has been removed, show that $w = f(z)$ maps the ring $\rho < |z| < \rho^{-1}$ onto the full w-plane from which the linear segment $-\alpha \le w \le \alpha$ and the rays $-\infty \le w \le -\alpha^{-1}$, $\alpha^{-1} \le w \le \infty$ have been removed. Show further that $f(z)$ can be continued analytically beyond $|z| = 1$ by the relation

$$f(z)\overline{f\left(\frac{1}{\bar{z}}\right)} = 1,$$

while its continuation beyond $|z| = \rho$ is given by

$$f(z) = \overline{f\left(\frac{\rho^2}{\bar{z}}\right)}.$$

5. If $w = f(z)$ maps the rectangle with the corners $-a, a, a + bi, -a + bi$ $(a, b > 0)$ onto the half-plane $\text{Im}\ \{w\} > 0$ and if $f(-a) = \alpha, f(a) = \beta, \alpha < \beta$, show that $w = f(z)$ maps the rectangle with the corners $-a - bi, a - bi, a + bi, -a + bi$ onto the full w-plane from which the rays $-\infty \le w \le \alpha$, $\beta \le w \le \infty$ have been removed. If $\alpha > \beta$, show that the map of the large rectangle is the full w-plane from which the linear segment $\beta \le w \le \alpha$ has been removed.

6. Let the function $f(z)$ be regular in a domain D whose boundary includes an analytic arc α. If the limits for $z \to \alpha$ of either of the expressions

$$\text{Re}\ \{f(z)\}, \quad \text{Im}\ \{f(z)\}, \quad |f(z) - \gamma|,$$

where γ is a constant, are the same at all points of α, show that $f(z)$ is regular on α.

7. If the domain D is bounded by n closed analytic curves $C_1, \ldots, C_n$, and if $\omega_\nu(z)$ denotes the harmonic measure of C_ν $(\nu = 1, \ldots, n)$ defined in Sec. 10, Chap. I, show that $\omega_\nu(z)$ is harmonic at all points of the boundary curves $C_1, \ldots, C_n$.

6. The Schwarz-Christoffel Formula. While the Riemann mapping theorem assures us that any two simply-connected domains with more than one boundary point can be mapped conformally upon each other, it is not of much help when we are faced with the practical problem of finding the mapping function which transforms two given domains into each other. The necessity thus arises of developing special techniques which will help us in the treatment of a given mapping problem. It is obviously sufficient to adapt these techniques to the case in which one of the two domains is a circle; if we can map two domains onto the same circle, we can also map them onto each other. The choice of the circle as the *standard domain*, or the *canonical domain*, in the simply-connected case has the advantage of leading to comparatively simple formulas. We shall also employ a half-plane in the capacity of a canonical domain if this results in even greater simplification. Since a circle and a half-plane are transformed into each other by a linear substitution, the mapping formulas involving these two domains can be easily transformed into each other.

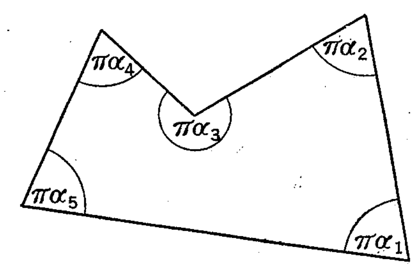

FIG. 13.

While it is in the nature of things that a simple solution of the general mapping problem cannot be expected, there are many important cases in which the mapping functions can be found by means of comparatively simple devices. In this section we shall treat the conformal mapping of a general polygon onto a circle or a half-plane.

Let then D be a polygon in the w-plane and let $\pi\alpha_1, \pi\alpha_2, \ldots, \pi\alpha_n$ denote its interior angles (see Fig. 13 for the case $n = 5$). In the formulas, it is more convenient to use the exterior angles $\pi\mu_1, \ldots, \pi\mu_n$, defined by $\pi\alpha_\nu + \pi\mu_\nu = \pi$, $\nu = 1, \ldots, n$. It is clear that $\mu_\nu > 0$ corresponds to a projecting corner and that $\mu_\nu < 0$ corresponds to the opposite case. If the polygon is convex, then all numbers μ_ν are positive. By a theorem of elementary geometry (which is almost self-evident), the sum of all exterior angles of a closed polygon is 2π. Hence, the quantities μ_ν introduced above are connected by the relation

$$\sum_{\nu=1}^{n} \mu_\nu = 2. \tag{46}$$

Let now $w = f(z)$ be an analytic function that maps the upper half-plane $\text{Im}\ \{z\} > 0$ onto the interior of the polygon D and let the points a_1,

$a_2, \ldots, a_n$ in the z-plane correspond to the vertices of the polygon whose exterior angles are $\pi\mu_1, \pi\mu_2, \ldots, \pi\mu_n$.

The points $a_1, \ldots, a_n$ divide the real axis into n parts each of which is mapped by $w = f(z)$ onto a linear segment. By the symmetry principle, $f(z)$ is therefore regular at all points of the real axis except at the points $a_1, \ldots, a_n$, and across each of the intervals bounded by these points $f(z)$ can be continued analytically by simple symmetry. The mirror image D' of D with respect to one of its sides will thus be the conformal map of the lower half-plane Im $\{z\} < 0$. Applying the symmetry principle again, this time with respect to one of the sides of D', we find that $w = f(z)$ maps the upper half-plane onto a figure D'' that is congruent with D but has a different location in the w-plane (see Fig. 14). These two inversions return a point z into its original position, while a point w is made subject to a translation and a rotation about the origin. The value $f_1(z)$ of the function $f(z)$ with which we return is therefore of the form

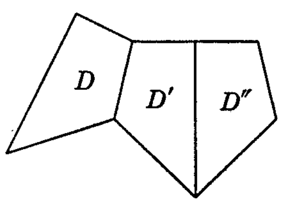

FIG. 14.

$$f_1(z) = Af(z) + B, \tag{46'}$$

where A and B are constants. It should be noted that the singularities of $f_1(z)$ are also situated at the points $a_1, \ldots, a_n$; obviously, these points are not affected by inversions with respect to the real axis. From (46′), we obtain

$$\frac{f_1''(z)}{f_1'(z)} = \frac{f''(z)}{f'(z)}.$$

This shows that the function

$$g(z) = \frac{f''(z)}{f'(z)} \tag{47}$$

returns to its initial value if z returns to its initial position by means of two inversions with respect to the real axis. Since, clearly, all possible branches of $f(z)$ are obtained from the initial branch by an even number of inversions (all images of the upper half-plane are polygons congruent to D and all images of the lower half-plane are congruent to the mirror image of D), it follows that the function $g(z)$ defined in (47) is single-valued in the whole z-plane. The singularities of $g(z)$ can only be located at the points $a_1, \ldots, a_n$.

In order to determine the character of these singularities, we consider the behavior of $f(z)$ in the neighborhood of the point a_ν; for the time being, we assume that none of the points $a_1, \ldots, a_n$ coincide with $z = \infty$. A small section of the real axis containing a_ν is mapped by $w = f(z)$ onto two linear segments which intersect at $w = f(a_\nu)$ with the angle $\pi\alpha_\nu$. Consider

now the function

$$h(z) = [f(z) - f(a_\nu)]^{\frac{1}{\alpha_\nu}} \tag{48}$$

near the point $z = a_\nu$. Since the mapping $z \to z^\gamma$ $(\gamma > 0)$ transforms rays emanating from the origin into rays emanating from the origin but multiplies all angles with vertex at the origin by the factor γ, it follows that $h(z)$ maps a small section of the real axis containing a_ν into two linear segments forming the angle π. In other words, the two segments belong to the same straight line. Thus, the function maps a linear segment containing a_ν onto a linear segment; by the symmetry principle, it is therefore regular at $z = a_\nu$. With $\alpha_\nu = 1 - \mu_\nu$, it thus follows from (48) that $f(z)$ is of the form

$$f(z) = f(a_\nu) + [h(z)]^{1-\mu_\nu},$$

where $h(z)$ is regular at $z = a_\nu$ and $h(a_\nu) = 0$, $h'(a_\nu) \neq 0$. $h(z)$ may also be written in the form $h(z) = (z - a_\nu)h_1(z)$, where $h_1(z)$ is regular at $z = a_\nu$ and $h_1(a_\nu) \neq 0$. Using this, we obtain

$$f(z) = f(a_\nu) + (z - a_\nu)^{1-\mu_\nu}[h_1(z)]^{1-\mu_\nu},$$

whence

$$\frac{f''(z)}{f'(z)} = -\frac{\mu_\nu}{(z - a_\nu)} + k(z),$$

where $k(z)$ is regular at $z = a_\nu$ and the fact that $h_1(a_\nu) \neq 0$ has been used. Comparison with (47) shows that the function

$$g(z) + \frac{\mu_\nu}{z - a_\nu}$$

is regular at the point $z = a_\nu$. Carrying through the same procedure at all points $a_1, \ldots, a_n$, we thus find that the function

$$g_1(z) = g(z) + \sum_{\nu=1}^{n} \frac{\mu_\nu}{z - a_\nu} \tag{49}$$

is regular at all points $a_1, \ldots, a_n$. But these were the only singular points of the single-valued function $g(z)$. Hence, $g(z)$ is regular and single-valued in the entire plane (including $z = \infty$); by Liouville's theorem, it therefore reduces to a constant. Moreover, this constant must be zero. Indeed, $f(z)$ is regular at $z = \infty$ and has therefore an expansion $f(z) = f(\infty) + c_1z^{-1} + c_2z^{-2} + \cdots$ near $z = \infty$. Differentiating, we find that $f'(z)$ has a zero of second order at $z = \infty$ while $f''(z)$ has a zero of third order there. It therefore follows from (47) that $g(z)$

vanishes at $z = \infty$. Since $(z - a_\nu)^{-1}$ also vanishes there, we conclude from (49) that $g_1(\infty) = 0$. But $g_1(z)$ was shown to be a constant and it is therefore zero everywhere. Combining (47) and (49), we therefore obtain

$$\frac{f''(z)}{f'(z)} = -\sum_{\nu=1}^{n} \frac{\mu_\nu}{z - a_\nu}.$$

By integrating this expression, the desired mapping function is finally found to be

$$f(z) = \alpha \int_0^z \frac{dz}{(z - a_1)^{\mu_1}(z - a_2)^{\mu_2} \cdots (z - a_n)^{\mu_n}} + \beta, \tag{50}$$

where α and β are integration constants determining the position and size of the polygon.

The formula (50) holds if none of the points a_ν coincide with the point at infinity. However, this restriction can easily be removed by means of a linear transformation. If, for instance, we transform the point a_n into the point at infinity by the linear substitution $z = a_n - (1/\zeta)$, we obtain

$$f(\zeta) = \alpha \int_0^\zeta \left(a_n - a_1 - \frac{1}{\zeta}\right)^{-\mu_1} \cdots \left(a_n - a_{n-1} - \frac{1}{\zeta}\right)^{-\mu_{n-1}} \left(-\frac{1}{\zeta}\right)^{-\mu_n} \frac{d\zeta}{\zeta^2} + \beta_1,$$

whence, in view of (46),

$$f(z) = \alpha_1 \int_0^z \frac{dz}{(z - a_1')^{\mu_1} \cdots (z - a_{n-1}')^{\mu_{n-1}}} + \beta_1, \tag{51}$$

where $a_1', \ldots, a_{n-1}'$ are constants. Hence, the effect on (50) of one of the points a_ν coinciding with the point at infinity simply consists in the corresponding term being left out of the formula.

By the linear transformation

$$z = i\left(\frac{1+\zeta}{1-\zeta}\right), \qquad \zeta = \frac{z-i}{z+i}$$

which maps $|\zeta| < 1$ onto $\operatorname{Im}\{z\} > 0$, we can also obtain from (50) a formula for the conformal map of the unit circle onto the polygon D. We have

$$(z - a_\nu)^{\mu_\nu} = \left[i\left(\frac{1+\zeta}{1-\zeta}\right) - a_\nu\right]^{\mu_\nu} = \left(\frac{a_\nu + i}{1-\zeta}\right)^{\mu_\nu}\left[\zeta - \frac{a_\nu - i}{a_\nu + i}\right]^{\mu_\nu}$$

and

$$dz = \frac{2i\,d\zeta}{(1-\zeta)^2}.$$

If we denote by b_ν the point

$$b_\nu = \frac{a_\nu - i}{a_\nu + i}$$

on the unit circle which is mapped onto the vertex of index ν, it follows therefore from (46) and (50) that

$$f(z) = \alpha_2 \int_0^z \frac{dz}{(b_1 - z)^{\mu_1} \cdots (b_n - z)^{\mu_n}} + \beta_2, \tag{52}$$

where the variable has again been denoted by z, and α_2, β_2 are constants.

By the same procedure we can also find the mapping function of the exterior of the polygon D. Since the angles $\pi\alpha_\nu$ are now replaced by $2\pi - \pi\alpha_\nu = \pi(2 - \alpha_\nu)$, the quantities $\mu_\nu = 1 - \alpha_\nu$ have to be replaced by $1 - (2 - \alpha_\nu) = -(1 - \alpha_\nu) = -\mu_\nu$. In analogy to (49), the function

$$g_2(z) = \frac{f''(z)}{f'(z)} - \sum_{\nu=1}^{n} \frac{\mu_\nu}{z - a_\nu} \tag{53}$$

will therefore be regular at the points $a_1, \ldots, a_n$. However, since the conformal map now contains the point at infinity, we cannot conclude without further investigation that $g_2(z)$ is a constant. Confining ourselves to the case in which the original domain is the unit circle, and assuming that $f(0) = \infty$, we thus have to study the behavior of the function f''/f' at $z = 0$. Since the mapping by $w = f(z)$ is conformal at $z = 0$, the singularity of $f(z)$ at this point is a simple pole. Hence, $f(z)$ must be of the form

$$f(z) = \frac{f_1(z)}{z},$$

where $f_1(z)$ is regular at $z = 0$ and $f_1(0) \neq 0$. It follows that

$$\frac{f''(z)}{f'(z)} = -\frac{2}{z} + \frac{zf_1''(z)}{zf_1'(z) - f_1(z)}.$$

This shows that

$$\frac{f''(z)}{f'(z)} + \frac{2}{z}$$

is regular at $z = 0$ and vanishes there. Combining this with the properties of the function $g_2(z)$ of (53), we find that the function

$$\frac{f''(z)}{f'(z)} - \sum_{\nu=1}^{n} \frac{\mu_\nu}{z - a_\nu} + \frac{2}{z} - \lambda, \qquad \lambda = \sum_{\nu=1}^{n} \frac{\mu_\nu}{d_\nu}$$

is regular and single-valued at all points of the plane and has the value 0 at $z = 0$. Hence, it reduces to the constant zero. Setting $z = \infty$, we find that we must have $\lambda = 0$. We thus obtain this formula

$$(54) \qquad f(z) = \alpha \int_{z_0}^{z} (z - a_1)^{\mu_1} \cdots (z - a_n)^{\mu_n} \frac{dz}{z^2}, \qquad z_0 \neq 0,$$

for the analytic function mapping $|z| < 1$ onto the exterior of the given polygon. In using this formula it should be noted that μ_ν is the exterior angle with respect to the interior of the polygon. The actual angle of the conformal map of $|z| < 1$ at the point $f(a_\nu)$ is $\pi(1 + \mu_\nu)$ (see Fig. 15).

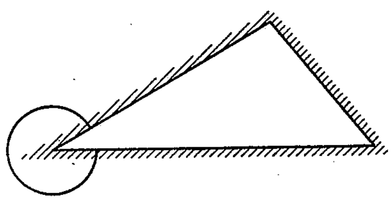

FIG. 15.

Formulas (50), (51), (52), and (54) are referred to as the *Schwarz-Christoffel formula.*

As an example for the use of the Schwarz-Christoffel formula we construct an analytic function $w = f(z)$ which maps the upper half-plane Im $\{z\} > 0$ onto the interior of a triangle of angles $\pi\alpha, \pi\beta, \pi\gamma$. As shown in the proof of the Riemann mapping theorem, the correspondence of three points on the boundaries of two simply-connected domains can be arbitrarily prescribed. We shall thus ask for the three vertices of the triangle to correspond to the points $z = 0, z = 1, z = \infty$. Under these conditions it follows from (51) that

$$(55) \qquad f(z) = C_1 \int_0^z z^{\alpha-1}(1 - z)^{\beta-1}\, dz + C_2.$$

From (55) we can easily find the lengths of the sides of the triangle. If a, b, c denote the sides of the triangle opposite the angles $\pi\alpha, \pi\beta, \pi\gamma$, respectively, and we set $C_1 = 1$, we have

$$c = \int_0^1 |f'(z)\, dz| = \int_0^1 |z^{\alpha-1}(1 - z)^{\beta-1}\, dz| = \int_0^1 \rho^{\alpha-1}(1 - \rho)^{\beta-1}\, d\rho = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)},$$

where $\Gamma(x)$ is the gamma function. In view of $\alpha + \beta + \gamma = 1$ and $\Gamma(x)\Gamma(1 - x) = \pi[\sin \pi x]^{-1}$, this can be brought into the form

$$c = \frac{1}{\pi} \sin \pi\gamma\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma).$$

Instead of evaluating a and b in a similar fashion, we can also observe that, by elementary trigonometry,

$$\frac{a}{\sin \pi\alpha} = \frac{b}{\sin \pi\beta} = \frac{c}{\sin \pi\gamma}.$$

Hence,

$$a = \frac{1}{\pi} \sin \pi\alpha\Gamma(\alpha)\Gamma(\beta)(\Gamma\gamma), \qquad b = \frac{1}{\pi} \sin \pi\beta\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma).$$

The determination of the vertices of the triangle for $C_1 = 1$, $C_2 = 0$ is left as an exercise to the reader.

The Schwarz-Christoffel formula remains correct if one of the corners of the polygon coincides with the point at infinity. As an example, we consider the function $w = f(z)$ which maps the upper half-plane Im $\{z\} > 0$ onto the interior of the "half-strip"

$$-\tfrac{1}{2}\pi < \text{Re } \{w\} < \tfrac{1}{2}\pi, \text{ Im } \{w\} > 0$$

(see Fig. 16) in such a way that the points $z = -1, 1, \infty$ and $w = -\frac{1}{2}\pi, \frac{1}{2}\pi, \infty$ correspond to each other in this order. Since the three angles of this "triangle" are $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, 0, we obtain from (51)

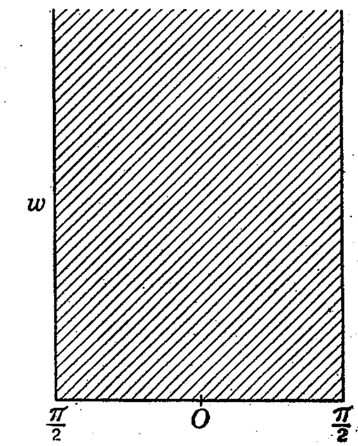

FIG. 16.

$$f(z) = \alpha_1 \int_0^z \frac{dz}{\sqrt{1 - z^2}} + \beta_1 = \alpha_1 \sin^{-1} z + \beta_1.$$

The constants α_1 and β_1 are determined by the conditions

$$-\tfrac{1}{2}\pi = f(-1) = -\tfrac{1}{2}\pi\alpha_1 + \beta_1, \qquad \tfrac{1}{2}\pi = f(1) = \tfrac{1}{2}\pi\alpha_1 + \beta_1.$$

It follows that $\beta_1 = 0$, $\alpha_1 = 1$, whence

$$w = f(z) = \sin^{-1} z.$$

We thus have the result that the inverse function $z = \sin w$ maps the half-strip of Fig. 16 onto the upper half-plane Im $\{z\} > 0$.

Finally, we consider the mapping $w = f(z)$ of the unit circle $|z| < 1$ onto the infinite strip $-\frac{1}{4}\pi < \text{Re } \{w\} < \frac{1}{4}\pi$ under the conditions

$$f(i) = f(-i) = \infty.$$

This infinite strip can be considered as a polygon with two sides which, at

$w = \infty$, form the angles 0. By (52), we obtain

$$f(z) = \alpha_2 \int_0^z \frac{dz}{(z - i)(z + i)} + \beta_2 = \alpha_2 \int_0^z \frac{dz}{1 + z^2} + \beta_2.$$

Hence, in view of $\tan^{-1}(\pm 1) = \pm\frac{1}{4}\pi$, it follows that $\alpha_2 = 1$, $\beta_2 = 0$ and, therefore,

$$f(z) = \tan^{-1} w.$$

The inverse function $w = \tan z$ thus maps the infinite strip $-\frac{1}{4}\pi < \text{Re}\,\{w\} < \frac{1}{4}\pi$ onto the unit circle $|z| < 1$. It should be pointed out that the use of the Schwarz-Christoffel formula in this degenerate case requires special justification and that it is easier to verify the result directly. The reader is recommended to do so by means of the known properties of the function $\tan w$.

EXERCISES

1. Show that

$$w = \int_0^z \frac{dz}{\sqrt{z(1 - z^2)}}$$

maps the upper half-plane $\text{Im}\,\{z\} > 0$ onto the interior of a square of side length

$$\frac{1}{2\sqrt{2\pi}}\,\Gamma^2\left(\frac{1}{4}\right).$$

2. Show that

$$w = \int_0^z \frac{dz}{\sqrt{1 - z^4}}$$

maps $|z| < 1$ onto the interior of a square of diagonal $[2\sqrt{2\pi}]^{-1}\Gamma^2(\frac{1}{4})$.

3. Show that

$$w = \int_{z_0}^z \frac{\sqrt{1 - z^4}}{z^2}\,dz, \qquad z_0 \neq 0,$$

maps $|z| < 1$ onto the exterior of a square.

4. Show that

$$w = \int_0^z \frac{dz}{(1 - z^n)^{\frac{2}{n}}}$$

maps $|z| < 1$ onto the inside of a regular polygon of order n whose side length is

$$\frac{1}{n}\,2^{1-\frac{4}{n}}\,\frac{\Gamma^2\left(\frac{1}{2} - \frac{1}{n}\right)}{\Gamma\left(1 - \frac{2}{n}\right)}.$$

5. Show that

$$w = \int_0^z \frac{(1 - z^5)^{\frac{2}{5}}}{(1 + z^5)^{\frac{4}{5}}}\,dz$$

maps the unit circle $|z| < 1$ onto the pentagram of Fig. 17. If R is the radius of the circumscribed circle, show that

$$R = \int_{-1}^0 (1 - t^5)^{\frac{2}{5}}(1 + t^5)^{-\frac{4}{5}}\,dt,$$

whence, by the substitution

$$u = \left(\frac{1 + t^5}{1 - t^5}\right)^2,$$

$$R = \frac{1}{5 \cdot 2^{\frac{2}{5}}} \int_0^1 u^{-\frac{9}{10}}(1 - u)^{-\frac{4}{5}}\,du$$

and thus

$$R = \frac{\Gamma(\frac{1}{10})\Gamma(\frac{1}{5})}{2^{\frac{2}{5}} \cdot 5\Gamma(\frac{3}{10})}.$$

6. Show that the function

$$w = \int_0^z \frac{dz}{(1 - z^4)\sqrt{1 + z^4}}$$

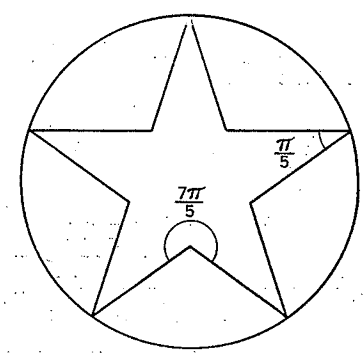

FIG. 17.

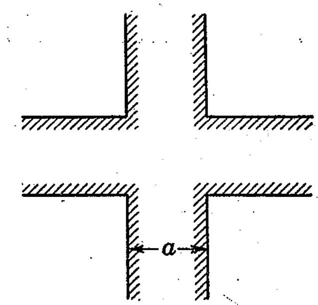

FIG. 18.

maps $|z| < 1$ onto the domain indicated in Fig. 18, and show that the width a is

$$a = \frac{1}{\sqrt{2}} \int_{-1}^1 \frac{dx}{(1 + x^4)\sqrt{1 - x^4}} = \tfrac{1}{8}[\pi\sqrt{2} + \pi^{-\frac{1}{2}}\Gamma^2(\tfrac{1}{4})].$$

7. Show that the function

$$w = \int_0^z \frac{1 - 2\cos 2\alpha z^2 + z^4}{(1 - z^2)(1 + z^2)^2}\,dz = \int_0^z \left[\frac{\sin^2\alpha}{1 - z^2} + \frac{(1 - z^2)\cos^2\alpha}{(1 + z^2)^2}\right]dz = \frac{1}{2}\sin^2\alpha \log\left(\frac{1 + z}{1 - z}\right) + \frac{z\cos^2\alpha}{1 + z^2}$$

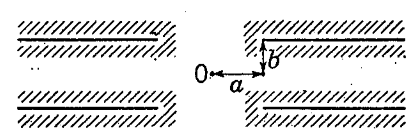

FIG. 19.

maps $|z| < 1$ onto the full w-plane which has been cut as indicated in Fig. 19. Show that the distances a and b are given by

$$a + ib = f(e^{i\alpha}) = \tfrac{1}{2}\sin^2\alpha \log\cot\tfrac{1}{2}\alpha + \tfrac{1}{2}\cos\tfrac{1}{2}\alpha + \tfrac{1}{2}\pi i \sin^2\alpha.$$

8. Using the symmetry principle, show that the function $w = \sin z$ maps the infinite strip $-\frac{1}{2}\pi < \text{Re}\{z\} < \frac{1}{2}\pi$ onto the full w-plane which has been cut along the two rays $-\infty \leq w \leq -1, 1 \leq w \leq \infty$.

9. Show that an analytic function $w = f(z)$ which maps the unit circle onto a convex polygon and satisfies $f(0) = 0, f'(0) = 1$ must be of the form

$$f(z) = \int_0^z \frac{dz}{\prod_{\nu=1}^{n} (1 - e^{i\theta_\nu} z)^{\mu_\nu}}, \qquad \mu_\nu > 0,\ 0 \leq \theta_\nu < 2\pi,$$

and deduce that $f(z)$ is subject to the inequalities

$$|f'(z)| \leq \frac{1}{(1-\rho)^2}, \qquad |f(z)| \leq \frac{\rho}{1-\rho}, \qquad |z| = \rho < 1.$$

10. Show that the entire w-plane which has been cut along the ray $-\infty \leq w \leq -\frac{1}{4}$ can be regarded as a polygon with the two vertices $w = -\frac{1}{4}$ and $w = \infty$ and the corresponding exterior angles $-\pi$ and 3π. Using the Schwarz-Christoffel formula, show then that the function $w = f(z)$ which maps $|z| < 1$ onto this cut plane and satisfies $f(-1) = -\frac{1}{4}, f(1) = \infty, f(0) = 0$, is of the form

$$w = f(z) = \frac{z}{(1-z)^2}.$$

7. Domains Bounded by Circular Arcs. In this section we shall consider the conformal mapping of domains which are bounded by a finite number of circular arcs. For greater brevity, such a domain will be referred to as a *curvilinear polygon* (Fig. 20). Our aim is to find the function $w = f(z)$ which maps the upper half-plane Re $\{z\} > 0$ onto the interior of this figure. In the similar problem of the preceding section, the crucial step was the introduction of the differential operator w''/w' which is not affected if the function w is replaced by $aw + b$, where a and b are arbitrary constants. To put it differently, this operator is invariant under a linear substitution which transforms any straight line into any other straight line. In the present problem, the domain in question is not bounded by linear segments but by circular arcs, and it may therefore be expected that a fundamental role will be played by a differential operator which is not susceptible to transformations carrying circles into circles, *i.e.*, general linear transformations.

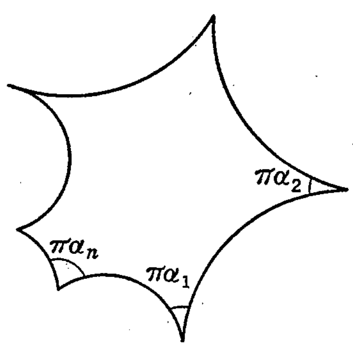

FIG. 20.

We shall show that the differential operator

$$\{w,z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2, \qquad w' = \frac{dw}{dz}, \tag{56}$$

which is known by the name of the *Schwarzian derivative* or the *Schwarzian differential parameter*, has precisely this property. In the notation (56), this amounts to showing that

$$\{W,z\} = \{w,z\}, \qquad W = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0, \tag{57}$$

where a, b, c, d are constants. The identity (57) can be confirmed by a formal computation. We have

$$W' = \frac{(ad - bc)}{(cw + d)^2} w',$$

whence, by logarithmic differentiation,

$$\frac{W''}{W'} = \frac{w''}{w'} - \frac{2cw'}{cw + d}.$$

Hence,

$$\left(\frac{W''}{W'}\right)' = \left(\frac{w''}{w''}\right)' + \frac{2c^2w'^2}{(cw + d)^2} - \frac{2cw''}{cw + d},$$

and

$$\left(\frac{W''}{W'}\right)^2 = \left(\frac{w''}{w'}\right)^2 + \frac{4c^2w'^2}{(cw + d)^2} - \frac{4cw''}{cw + d},$$

and therefore

$$\left(\frac{W''}{W'}\right)' - \frac{1}{2}\left(\frac{W''}{W'}\right)^2 = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2,$$

which, in view of (56), proves (57).

Returning now to the problem of mapping the upper half-plane onto the curvilinear polygon, we first observe that, in view of the symmetry principle, the mapping function $w = f(z)$ must be regular at all points of the real axis except at the points $a_1, a_2, \ldots, a_n$ which correspond to the vertices of the polygon. In view of the fact that the mapping $z \to f(z)$ is conformal at all points of Im $\{z\} > 0$ and at the points of the real axis other than a_ν, $\nu = 1, \ldots, n$, the derivative $f'(z)$ does not vanish there. It follows that the expression $\{w,z\}$ $[w = f(z)]$ is regular in the closed half-plane Im $\{z\} \geq 0$, with the exception of the points a_ν. We now use the invariance property (57) of the Schwarzian derivative. By a suitable linear transformation, any one of the circular arcs bounding our polygon can be mapped onto part of the real axis. If $w \to W$ is this linear transformation, it follows from (57) that $\{w,z\} = \{W,z\}$. Now the transformation $z \to W$ maps a part of the real axis—the linear segment $a_\nu < z < a_{\nu+1}$, say—onto another part of the real axis. For these values of z, the function $W = W(z)$ is a real function, and the same is therefore true of its derivatives. In view of the definition (56) of the Schwarzian derivative $\{W,z\}$, it follows that $\{W,z\}$ takes real values for $a_\nu < z < a_{\nu+1}$. Hence, $\{w,z\}$ is also real for $a_\nu < z < a_{\nu+1}$. We thus have proved that the Schwarzian derivative $\{w,z\}$ of the mapping function $w = f(z)$ is real at all points of the real axis except at the points $a_1, \ldots, a_n$.

At the points $a_1, \ldots, a_n$, the function $w = f(z)$—and therefore also its Schwarzian derivative—will have singularities. In order to study the nature of these singularities of $\{w,z\}$, we use again the invariance of $\{w,z\}$ with respect to an arbitrary linear transformation. Considering, in particular, the vertex corresponding to $z = a_\nu$, we can perform a linear transformation which carries this vertex into the origin and transforms the two circles meeting at this vertex with the angle $\pi\alpha_\nu$ (see Fig. 20) into two straight lines. Since the mapping is conformal, these two straight lines will meet at the origin with the same angle $\pi\alpha_\nu$. Now $\{w,z\}$ was not affected by this linear transformation; hence, the singularity of the function $\{w,z\}$ at $z = a_\nu$ can be obtained from the assumption that the function $w = f(z)$ maps a piece of the real axis containing $z = a_\nu$ onto two linear segments meeting at the origin with the angle $\pi\alpha_\nu$. As shown in the preceding section, such a function $f(z)$ is of the form

$$f(z) = (z - a_\nu)^{\alpha_\nu} f_1(z),$$

where $f_1(z)$ is regular at $z = a_\nu$, $f_1(a_\nu) \neq 0$, and $f_1(z)$ is a real function if z is real. Using this representation, we obtain, by an elementary computation,

$$\{w,z\} = \left[\frac{f''(z)}{f'(z)}\right]' - \frac{1}{2}\left[\frac{f''(z)}{f'(z)}\right]^2 = \frac{1}{2}\frac{1-\alpha_\nu^2}{(z-a_\nu)^2} + \frac{\beta_\nu}{z-a_\nu} + f_2(z),$$

where $f_2(z)$ is regular at $z = a_\nu$ and

$$\beta_\nu = \frac{1-\alpha_\nu^2}{\alpha_\nu}\frac{f_1'(a_\nu)}{f_1(a_\nu)}$$

is real. By applying the same procedure to all the points $a_1, \ldots, a_n$, we thus find that the expression

$$\{w,z\} - \frac{1}{2}\sum_{\nu=1}^{n}\frac{1-\alpha_\nu^2}{(z-a_\nu)^2} - \sum_{\nu=1}^{n}\frac{\beta_\nu}{z-a_\nu}$$

is regular at the points $a_1, \ldots, a_n$ and, therefore, at all points of the real axis. This expression is, moreover, real at all points of the real axis. Indeed, $\{w,z\}$ was shown above to be real for real z, and the reality of the terms involving $(z - a_\nu)^{-1}$ and $(z - a_\nu)^{-2}$ follows from the fact that the constants α_ν, β_ν, a_ν are real. But, as shown in Sec. 10, Chap. III, an analytic function which is regular in the closure of a domain and takes real values on the boundary reduces to a constant. Hence, *the function $w = f(z)$ mapping* Im $\{z\} > 0$ *onto a curvilinear polygon with angles $\pi\alpha_1, \ldots, \pi\alpha_n$ satisfies the differential equation*

$$\text{(58)} \qquad \{w,z\} = \frac{1}{2}\sum_{\nu=1}^{n}\frac{1-\alpha_\nu^2}{(z-a_\nu)^2} + \sum_{\nu=1}^{n}\frac{\beta_\nu}{z-a_\nu} + \gamma,$$

where $\beta_1, \ldots, \beta_n, \gamma$ are real constants.

The constants γ and β_ν are, however, not entirely independent of each other. If none of the points $a_1, \ldots, a_n$ coincide with the point at infinity, $w = f(z)$ must be regular at $z = \infty$. Hence, there is an expansion

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

which converges near $z = \infty$. Inserting this in (56), we find by a formal computation that the expansion of $\{w,z\}$ near $z = \infty$ starts with the term in z^{-4}. Since the first terms of the corresponding expansion of the right-hand side of (58) are

$$\gamma + \frac{1}{z}\sum_{\nu=1}^{n}\beta_\nu + \frac{1}{z^2}\sum_{\nu=1}^{n}\left[a_\nu\beta_\nu + \frac{1}{2}(1-\alpha_\nu^2)\right] + \frac{1}{z^3}\sum_{\nu=1}^{n}[\beta_\nu a_\nu^2 + a_\nu(1-\alpha_\nu^2)] + \cdots,$$

it follows that the conditions

$$\text{(59)} \qquad \gamma = 0, \qquad \sum_{\nu=1}^{n}\beta_\nu = 0, \qquad \sum_{\nu=1}^{n}[2a_\nu\beta_\nu + 1 - \alpha_\nu^2] = 0, \qquad \sum_{\nu=1}^{n}[\beta_\nu a_\nu^2 + a_\nu(1-\alpha_\nu^2)] = 0$$

must be satisfied. The reader will confirm without difficulty that the first two conditions (59) also hold if one of the vertices of the polygon corresponds to $z = \infty$ and that, in this case, the expansion of $\{w,z\}$ near

$z = \infty$ starts with the term $\frac{1}{2}(1 - \alpha^2)z^{-2}$, if $\pi\alpha$ is the corresponding angle of the polygon.

It is easy to see that the four identities (59) are the only general relations which can exist between the constants entering the equation (58). The solutions of (58) must be able to represent the most general curvilinear polygon P_n whose sides are n circular arcs. Since a circle is determined by $3n$ real parameters (radius and coordinates of the center), there is a $3n$-parameter family of such polygons. From these we have to deduct six real parameters since the equation (58) determines P_n only up to an arbitrary linear transformation depending on six real parameters (three arbitrary points can be made to correspond to three given points). This leaves $3n - 6$ independent real parameters determining a polygon P_n. In (58), there appear explicitly $3n + 1$ parameters, namely, γ and the constants α_ν, a_ν, β_ν. Deducting from these the four relations (59), we still have a balance of $3n - 3$, three more than we need. However, this excess of parameters is only apparent. By a linear transformation of the upper half-plane onto itself, three of the points $a_1, \ldots, a_n$ can be brought into prescribed positions on the real axis, thus eliminating another triple of constants entering (58). The number of these constants is thus reduced to $3n - 6$. Since this is precisely the number of parameters characterizing a polygon P_n, it follows that no more relations between the constants in (58) can be expected.

The difficulty in constructing the mapping function of a given curvilinear polygon from the differential equation (58) is due not so much to the fact that we have to integrate a differential equation of the third order; we shall see presently that our task can be reduced to the integration of a comparatively simple linear differential equation of the second order. The real difficulty is caused by the fact that the connection between the constants entering (58)—excepting, of course, the α_ν which are given by the angles—and the geometric configuration of the polygon P_n is extremely unobvious. $n - 3$ of these constants can be determined by "non-Euclidean" conditions, namely, by prescribing the points a_ν on the real z-axis which are to correspond by the mapping $w = f(z)$ to the vertices of P_n (it has been mentioned before that three of the points a_ν are arbitrary). Deducting further the n constants α_ν which are given by the angles of P_n, we are thus left with $n - 3$ constants, the so-called *accessory parameters*, whose determination by means of geometric conditions, whether Euclidean or non-Euclidean, is an extremely difficult task. Except for the case $n = 2$ (which can be treated by much more elementary means), the only case which is free of accessory parameters is that of a curvilinear triangle.

In this case, all constants entering the equation (58) can be expressed in terms of the given quantities. If we write $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$,

$a_1 = a$, $a_2 = b$, $a_3 = c$, it follows from (59) and an elementary computation that the equation (58) reduces in this case to

$$(60) \quad \{w,z\} = \frac{1}{(z-a)(z-b)(z-c)}\left[\frac{1-\alpha^2}{2}\frac{(a-b)(a-c)}{z-a} + \frac{1-\beta^2}{2}\frac{(b-a)(b-c)}{z-b} + \frac{1-\gamma^2}{2}\frac{(c-a)(c-b)}{z-c}\right].$$

This expression can be simplified by identifying a, b, c with the points $z = 0$, $z = \infty$, $z = 1$, respectively. If we let $b \to \infty$ in (60), the expressions

$$\frac{a-b}{z-b}, \qquad \frac{(b-a)(b-c)}{(z-b)^2}, \qquad \frac{c-b}{z-b}$$

tend to 1 and we obtain

$$\{w,z\} = \frac{1}{(z-a)(z-c)}\left[\frac{1-\alpha^2}{2}\frac{a-c}{z-a} + \frac{1-\beta^2}{2} + \frac{1-\gamma^2}{2}\frac{c-a}{z-c}\right].$$

Hence, for $a = 0$, $c = 1$,

$$\{w,z\} = \frac{1}{z(z-1)}\left[-\frac{1-\alpha^2}{2z} + \frac{1-\beta^2}{2} + \frac{1-\gamma^2}{2(z-1)}\right],$$

which can also be brought into the form

$$(61) \qquad \{w,z\} = \frac{1-\alpha^2}{2z^2} + \frac{1-\gamma^2}{2(z-1)^2} + \frac{\alpha^2+\gamma^2-\beta^2-1}{2z(z-1)}.$$

However, before we enter into a further discussion of (61), we have to examine the differential equation (58) in greater detail. If w is a solution of (58), the same is true of the function W defined in (57) which contains three arbitrary constants (one of the four constants a, b, c, d can be made equal to 1 without altering the value of W). Since, on the other hand, (58) is a differential equation of the third order whose general solution cannot contain more than three independent arbitrary constants, it is thus sufficient to find one solution of (58); the general solution will then follow by an arbitrary linear transformation. The finding of one particular solution is further facilitated by a connection existing between an equation of the type (58) and a linear differential equation of the second order. The result to which we are referring is the following. *If u_1 and u_2 are two linearly independent solutions of the linear differential equation*

$$u''(z) + p(z)u(z) = 0,$$

then

$$w(z) = \frac{u_1(z)}{u_2(z)}$$

is a solution of the equation

(62) $$\{w,z\} = 2p(z).$$

The truth of this result is easily confirmed. Substituting $u_2 w$ for u_1 in the equation $u_1'' + pu_1 = 0$, we obtain

$$u_2 w'' + 2u_2'w' + w(u_2'' + pu_2) = 0$$

and therefore, in view of $u_2'' + pu_2 = 0$, $u_2 w'' + 2u_2'w' = 0$. Hence,

$$\frac{w''}{w'} = -2\frac{u_2'}{u_2},$$

and thus

$$\left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = -2\left(\frac{u_2'}{u_2}\right)' - 2\left(\frac{u_2'}{u_2}\right)^2 = -\frac{2u_2''}{u_2}.$$

In view of $u_2'' + pu_2 = 0$ and (56), this is equivalent to (62).

Combining (62) with (58) and (59), we thus obtain the following result.

If $w = f(z)$ maps the upper half-plane Im $\{z\} > 0$ *onto a curvilinear polygon composed of n circular arcs and if the point $z = a_\nu$ on the real axis corresponds to a vertex of angle $\pi\alpha_\nu$, then*

(63) $$w = f(z) = \frac{u_1(z)}{u_2(z)},$$

where $u_1(z)$ and $u_2(z)$ are two linearly independent solutions of the linear differential equation

(64) $$u''(z) + \left[\frac{1}{4}\sum_{\nu=1}^{n}\frac{1-\alpha_\nu^2}{(z-a_\nu)^2} + \frac{1}{2}\sum_{\nu=1}^{n}\frac{\beta_\nu}{z-a_\nu}\right]u(z) = 0$$

and the real constants β_ν are subject to the relations

(65) $$\sum_{\nu=1}^{n}\beta_\nu = 0, \qquad \sum_{\nu=1}^{n}[2a_\nu\beta_\nu + 1 - \alpha_\nu^2] = 0,$$

$$\sum_{\nu=1}^{n}[\beta_\nu a_\nu^2 + a_\nu(1-\alpha_\nu^2)] = 0.$$

Since a differential equation of the type (64) is easily solved in terms of power series expansions, our mapping problem is therefore to be regarded

as solved if the constants β_ν are known. However, as already pointed out, the determination of the $n - 3$ independent constants β_ν in terms of the geometrical configuration of the curvilinear polygon is an exceedingly difficult task.

A complete treatment is possible in the case of a curvilinear triangle. If $\pi\alpha$, $\pi\beta$, $\pi\gamma$ are the angles of the triangle and $z = 0, \infty, 1$ the points corresponding to the vertices, it follows from (61) that the differential equation (64) takes the form

(66) $$u'' + \frac{1}{4}\left[\frac{1-\alpha^2}{z^2} + \frac{1-\gamma^2}{(z-1)^2} + \frac{\alpha^2+\gamma^2-\beta^2-1}{z(z-1)}\right]u = 0.$$

Since we are interested not in the individual solutions of this equation but in the quotient (63) of two solutions, we may replace (66) by a differential equation of the type

(67) $$y'' + P(z)y' + Q(z)y = 0$$

whose solutions are related to those of (66) by the identity

(68) $$y(z) = \sigma(z)u(z),$$

where $\sigma(z)$ is a given function. If $y_1(z)$ and $y_2(z)$ are two linearly independent solutions of (67), we clearly have

$$\frac{y_1(z)}{y_2(z)} = \frac{u_1(z)}{u_2(z)},$$

where $u_1(z)$ and $u_2(z)$ are linearly independent solutions of (66). Taking the function $\sigma(z)$ in (68) to be of the form

$$\sigma(z) = e^{-\frac{1}{2}\int^z P(z)\,dz},$$

we find by an elementary computation that the equation (67) is equivalent to the equation

$$u'' + [Q - \tfrac{1}{4}P^2 - \tfrac{1}{2}P']u = 0$$

for the function $u = u(z)$ defined in (68). A comparison with (66) shows therefore that the equations (66) and (67) will be equivalent (for our purposes) if the relation

(69) $$Q - \frac{1}{4}P^2 - \frac{1}{2}P' = \frac{1}{4}\left[\frac{1-\alpha^2}{z^2} + \frac{1-\gamma^2}{(z-1)^2} + \frac{\alpha^2+\gamma^2-\beta^2-1}{z(z-1)}\right]$$

is satisfied. We leave it as an exercise to the reader to show that (69) holds if

$$P(z) = \frac{c - (a + b + 1)z}{z(1 - z)}, \qquad Q(z) = -\frac{ab}{z(1 - z)},$$

where the constants a, b, c are defined by

$$(70) \qquad a = \tfrac{1}{2}(1 + \beta - \alpha - \gamma), \qquad b = \tfrac{1}{2}(1 - \alpha - \beta - \gamma), \qquad c = 1 - \alpha.$$

With these values of $P(z)$ and $Q(z)$, the equation (67) can be brought into the form

$$(71) \qquad z(1 - z)y'' + [c - (a + b + 1)z]y' - aby = 0.$$

This differential equation is known as the *hypergeometric equation* and plays an important part in many branches of pure and applied mathematical analysis. The properties of its solutions have been thoroughly investigated and have been made the subject of an extensive literature. Those properties of the solutions of (71) which are relevant from our present point of view will be found in the chapter on the hypergeometric function in Whittaker-Watson's "Modern Analysis," to which the reader is referred.

Summing up, we thus have the following result: *The function* $w = f(z)$ *which maps the upper half-plane* Im $\{z\} > 0$ *onto the interior of a curvilinear triangle with the angles* $\pi\alpha$, $\pi\beta$, $\pi\gamma$ *is of the form*

$$f(z) = \frac{y_1(z)}{y_2(z)},$$

where $y_1(z)$ *and* $y_2(z)$ *are two linearly independent solutions of the hypergeometric equation* (71) *and the constants* a, b, c *in* (71) *are related to* α, β, γ *by* (70).

The equation (71) is solved by the hypergeometric series

$$(72) \quad F(a,b,c;z) = 1 + \frac{ab}{c}z + \frac{a(a + 1)b(b + 1)}{c(c + 1)2!}z^2 + \frac{a(a + 1)(a + 2)b(b + 1)(b + 2)}{c(c + 1)(c + 2)3!}z^3 + \cdots, \qquad |z| < 1,$$

as the reader will verify without difficulty. The function $F(a,b,c;z)$ can also be represented in the form of a definite integral. We have

$$(73) \qquad F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a}\,dt,$$

where the conditions $b > 0$, $c > b$ are necessary for the existence of the integral. The identity of (73) and the series (72) is easily established by expanding $(1 - zt)^{-a}$ by powers of z and integrating term by term. In

view of (70), the conditions $b > 0$, $c > b$ are equivalent to $\alpha + \beta + \gamma < 1$, $\alpha < 1 + \beta + \gamma$; both will therefore be satisfied if the sum of the three angles of the curvilinear triangle is smaller than π. The integral representation (73) has many advantages over the infinite series (72). While the use of (72) is restricted to values of z such that $|z| < 1$, no such restriction applies to (73). (73) may therefore be used for all values of z in the upper half-plane. Besides, the convergence of the series (72) is very slow unless $|z|$ is small, while the value of the integral (73) can be easily computed with great accuracy.

For the solution of our mapping problem we need yet another solution of the equation (71). Such a solution is easily obtained from the observation that the substitution of $1 - z$ for z transforms (71) into

$$z(1 - z)y'' + [a + b - c + 1 - (a + b + 1)z]y' - aby = 0,$$

which is another hypergeometric equation. The parameters of this equation are $a_1 = a$, $b_1 = b$, $c_1 = a + b - c + 1$. A glance at (73) shows that this equation is solved by

$$(74) \qquad y = \int_0^1 t^{b-1}(1 - t)^{a-c}(1 - zt)^{-a}\,dt,$$

where the conditions $b > 0$ and $a > c - 1$ are required for the existence of the integral. These conditions are identical with $\alpha + \beta + \gamma < 1$, $\gamma - \beta - \alpha < 1$, and are therefore satisfied if the sum of the angles is smaller than π. If, in (74), z is again replaced by $1 - z$, we obtain a solution of the equation (71); the confirmation that (74), after this substitution, is not a constant multiple of (73) is left as an exercise to the reader. Our result concerning the mapping function of the curvilinear triangle takes thus the following explicit form.

The function

$$(75) \quad w = f(z) = \frac{\int_0^1 t^{-\frac{1}{2}(1+\alpha+\beta+\gamma)}(1 - t)^{-\frac{1}{2}(1+\alpha-\beta-\gamma)}(1 - zt)^{-\frac{1}{2}(1-\alpha+\beta-\gamma)}\,dt}{\int_0^1 t^{-\frac{1}{2}(1+\alpha+\beta+\gamma)}(1 - t)^{-\frac{1}{2}(1-\alpha-\beta+\gamma)}(1 - t + zt)^{-\frac{1}{2}(1-\alpha+\beta-\gamma)}\,dt}$$

maps the upper half-plane Im $\{z\} > 0$ *onto a curvilinear triangle with the angles* $\pi\alpha$, $\pi\beta$, $\pi\gamma$, *provided the sum of the angles is smaller than* π.

If $\alpha + \beta + \gamma = 1$, the triangle can be made rectilinear by a suitable linear transformation and the mapping function can be constructed by means of the Schwarz-Christoffel formula. If $\alpha + \beta + \gamma > 1$, then (73) and (74) have to be replaced by integral representations of the hypergeometric function which converge for these values of the parameters. The interested reader will find integral representations of this type in the book of Whittaker and Watson mentioned above.

A further discussion of the function $f(z)$ defined in (75) will be found in Sec. 5, Chap. VI.

EXERCISES

1. Using (75) and the identities

$$\int_0^1 t^{r-1}(1-t)^{s-1}\,dt = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \qquad r > 0,\ s > 0,$$

$$\Gamma(r)\Gamma(1-r) = \frac{\pi}{\sin \pi r},$$

show that the vertices with the angles $\pi\alpha$ and $\pi\gamma$ (corresponding to $z = 0$ and $z = 1$, respectively) are situated at the points

$$w = \frac{\sin \pi\alpha}{\cos \frac{\pi}{2}(\alpha - \beta - \gamma)}$$

and

$$w = \frac{\cos \frac{\pi}{2}(\alpha + \beta - \gamma)}{\sin \pi\gamma},$$

respectively.

2. If $\alpha = \gamma$, show that the function (75) satisfies the relation $f(z)f(1-z) = 1$. Use your result to prove that $w = f(z)$ maps the straight line $\text{Re}\,\{z\} = \frac{1}{2}$ onto part of the circumference $|w| = 1$. *Hint:* Use the fact that—for suitable determinations of the powers under the integral signs—$f(z)$ is real for $0 < z < 1$, and apply the symmetry principle.

3. Show that in the case in which the three circles forming the curvilinear triangle are tangent to each other (Fig. 21), the mapping function (75) takes the form

$$f(z) = \frac{T(z)}{T(1-z)},$$

where

$$T(z) = \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-zt)}}.$$

Show further that this is equivalent to

$$f(z) = \frac{K(z)}{K(1-z)},$$

where

$$K(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}}.$$

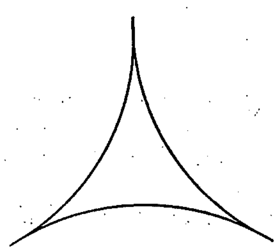

FIG. 21.

4. Show that the equation (58) for the function $w = f(z)$ which maps $\text{Im}\,\{z\} > 0$ onto the interior of a crescent-shaped (or lense-shaped, as the case may be) figure of angle α (see Fig. 22) is

$$\{w,z\} = \frac{(1-\alpha^2)(a-b)^2}{2(z-a)^2(z-b)^2},$$

where a and b are the points on the real axis corresponding to the vertices. Show further that the associated linear differential equation

$$u'' + \frac{(1-\alpha^2)(a-b)^2}{4(z-a)^2(z-b)^2}\,u = 0$$

is equivalent to the differential equation

$$y'' + (1-\alpha)\left[\frac{1}{z-a} + \frac{1}{z-b}\right]y' - \frac{\alpha(1-\alpha)}{(z-a)(z-b)}\,y = 0.$$

Verify that the latter equation has the solutions $y = (z-a)^\alpha$ and $y = (z-b)^\alpha$ and deduce that the mapping function is of the form

$$f(z) = \frac{A(z-a)^\alpha + B(z-b)^\alpha}{C(z-a)^\alpha + D(z-b)^\alpha},$$

where A, B, C, D are complex constants for which $AD - BC \neq 0$.

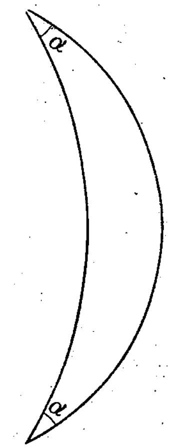

FIG. 22.

8. Univalent Functions. Of especial importance from the point of view of conformal mapping are those analytic functions $f(z)$ which are univalent in a given domain D. We recall that a univalent function in D is characterized by the fact that it takes in D no value more than once and that, consequently, it maps D onto a schlicht domain, *i.e.*, a domain which is not self-overlapping and contains no branch points. For the latter reason, univalent functions are also often referred to as *schlicht functions*. In the present section, we shall investigate some of the properties of analytic functions which are univalent in a given simply-connected domain D. We may, without an essential restriction of the generality of our considerations, confine ourselves to the case in which D is the unit circle. Indeed, by the Riemann mapping theorem, any simply-connected domain can be mapped onto the unit circle; accordingly, any univalent function in D is associated with a univalent function in the unit circle and the properties of the latter function can be easily translated into properties of the original function if the function mapping D onto the unit circle is known. The choice of the unit circle as the domain of definition of a univalent function has the advantage of simplifying the computations and of leading to short and elegant formulas.

A function $f(z)$ which is regular and univalent in the unit circle may further be normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Indeed, if $f(z)$ is univalent, so is the function

$$f_1(z) = \frac{f(z) - f(0)}{f'(0)},$$

and any property of the function $f_1(z)$ is immediately translated into a corresponding property of $f(z)$; we add that the division by $f'(0)$ is per-

14. Show that the transformation

$$w = \sin\left[\frac{\pi z}{1+z^2}\right]$$

maps the unit circle $|z| < 1$ onto a domain D of the following description: D covers the entire w-plane on infinity of times, with the exception of the point $w = \infty$ and the linear segment $-1 \leq w \leq 1$; the points $w = \infty$ and $w = \pm 1$ are not covered at all and the segment $-1 < w < 1$ is covered by D exactly once.

3. Elliptic Functions. In the two preceding sections we were studying the conformal mapping properties of analytic functions which were known a priori. In this section we shall adopt a somewhat different procedure. Since it is assumed that the reader is not acquainted with the theory of elliptic functions, we shall define some of the fundamental elliptic functions by means of certain conformal mappings effected by them, and we shall then proceed to derive some of their other properties.

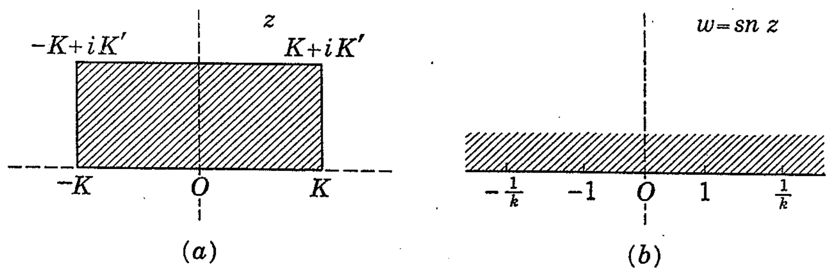

FIG. 34.

Our point of departure is the conformal mapping of the half-plane $\text{Im}\,\{w\} > 0$ onto a rectangle in the z-plane, where the points $w = \pm 1$, $w = \pm(1/k)$ $(0 < k < 1)$ are to correspond to the corners of the rectangle. By the Schwarz-Christoffel formula (50) of Sec. 6, Chap. V, this mapping is effected by the function

$$z = F(w) = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}. \tag{13}$$

In order to determine the position of the rectangle in the z-plane, we observe that $F(w)$ is real for real w and that $F(-w) = -F(w)$. It follows that one of the sides of the rectangle coincides with part of the real axis and is situated symmetrically with respect to the origin. If we agree to take in (13) that branch of the square root for which $\sqrt{1} = 1$ and denote the height and width of the rectangle by K' and $2K$, respectively, the position of the rectangle will therefore be as indicated in Fig. 34*a*.

We now define the elliptic function $w = \text{sn}\,z$ as the analytic function for which $\text{sn}'(0) = 1$ and which maps the rectangle of Fig. 34*a* onto the upper half-plane indicated in Fig. 34*b* in such a way that the points

(14)

z	$-K+iK'$	$-K$	0	K	$K+iK'$	iK'
sn z	$-k^{-1}$	-1	0	1	k^{-1}	∞

correspond to each other. We might also have defined $w = \text{sn}\,z$ as the inverse of the function $z = F(w)$ introduced in (13). Obviously, the function $w = \text{sn}\,z$ depends not only on z but also on the parameter k. It is, however, customary to regard one of the quantities

$$\tau = \frac{iK'}{K}, \tag{15}$$

$$q = e^{\pi i \tau} = e^{-\frac{K'}{K}}, \tag{16}$$

rather than k, as the parameter on which sn z depends. If it is desired to indicate the parameter, the symbol sn z is replaced by $\text{sn}(z,\tau)$ or $\text{sn}(z,q)$. We shall see later that k is uniquely determined if either τ or q are given.

The symbol sn z is used because of certain analogies between this function and the trigonometric function sin z. It is also easy to see that the function sin z corresponds to a degenerate case of the function sn z. This follows either by letting $k \to 0$ in (13) or by observing that, for $q \to 0$, the rectangle of Fig. 34*a* becomes an infinite half-strip. This analogy is carried further by the definition

$$\text{cn}\,z = \sqrt{1 - \text{sn}^2 z}, \qquad \text{cn}\,0 = 1, \tag{17}$$

of the elliptic function cn z. Another elliptic function is introduced by

$$\text{dn}\,z = \sqrt{1 - k^2\,\text{sn}^2 z}, \qquad \text{dn}\,0 = 1. \tag{18}$$

The functions sn z, cn z, dn z are referred to as the *Jacobian elliptic functions.* We add that the name "elliptic functions" is due to the fact that the integral (13) was first encountered in connection with the problem of finding the length of an arc of an ellipse. This led to the appellation "elliptic integrals" for a class of integrals related to (13) and, subsequently, to the name "elliptic functions" for the functions inverse to these integrals if the latter are regarded as functions of their upper limits.

With the help of (13) and (14), both K and K' can be expressed in terms of k. We have

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{19}$$

and

$$iK' = \int_1^{k^{-1}} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}.$$

The latter integral can be brought into a more elegant form. If we make the substitution

$$s = (1 - k'^2t^2)^{-\frac{1}{2}},$$

where

$$k' = \sqrt{1 - k^2}, \tag{20}$$

we obtain by an elementary computation

$$K' = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k'^2t^2)}}. \tag{21}$$

It is interesting to observe that the functional dependence of K with respect to k is the same as that of K' with respect to k', where k' is defined in (20), that is, $K'(k) = K(\sqrt{1 - k^2})$.

The fundamental property of the elliptic functions is their *double periodicity*, that is, the fact that such a function has two different periods which are not integral multiples of the same number. In the case of the function sn z, these periods are $4K$ and $2iK'$. In other words, the function sn z satisfies the identities

$$\begin{aligned} \text{sn}(z + 4K) &= \text{sn}\, z, \\ \text{sn}(z + 2iK') &= \text{sn}\, z. \end{aligned} \tag{22}$$

(22) can be proved by suitable application of the symmetry principle. Since $w = \text{sn}\, z$ maps the rectangle R of Fig. 35 onto the half-plane of Fig. 34*b*, sn z can be continued beyond the boundary of the rectangle by symmetry. Inverting the rectangle with respect to its upper side, we find that $w = \text{sn}\, z$ maps the rectangle R_1 of Fig. 35 onto the lower half-plane. Inverting, in turn, the rectangle R_1 with respect to its upper side, we see that the rectangle R_2 of Fig. 35 is again mapped by $w = \text{sn}\, z$ onto the upper half-plane. If z_1 and z_2 denote the points into which a point z of R is successively carried by these inversions, it is clear that $z_2 = z + 2iK'$. Since the image point of z is returned to its original position by the two inversions with respect to the real axis, it follows from the symmetry principle that $\text{sn}(z + 2iK') = \text{sn}\, z$. This proves the second identity (22). The proof of the first inequality (22) follows in the same way by considering the inversions R_3 and R_4 of R (see Fig. 35) and is left as an exercise to the reader.

Next, we show that *the function $w = \text{sn}\, z$ is single-valued at all finite points of the z-plane.* To this end, we divide the z-plane into a network of congruent rectangles by means of the lines Re $\{z\} = K(2n + 1)$, $n = 0, \pm1, \pm2, \ldots$ and Im $\{z\} = mK'$, $m = 0, \pm1, \pm2, \ldots$. All these rectangles can be obtained from the rectangle R of Fig. 35 by a suitable number of inversions. Although there are many different possibilities to get from R to a given rectangle R' in this fashion, a moment's reflection shows that the number of inversions connecting R and R' is always even if it is even for one particular chain of inversions, or else it is always odd if it is odd for one particular chain of inversions. On the other hand, an even number of inversions of a point w with respect to the real axis in the w-plane returns the point to its original position, while an odd number of such inversions carries the point w into the point $\bar{w}$. Hence, the analytic continuation of $w = \text{sn}\, z$ by means of the symmetry principle leads to a uniquely determined value of sn z at each point of the z-plane, regardless of the path along which sn z has been continued. This proves the above statement.

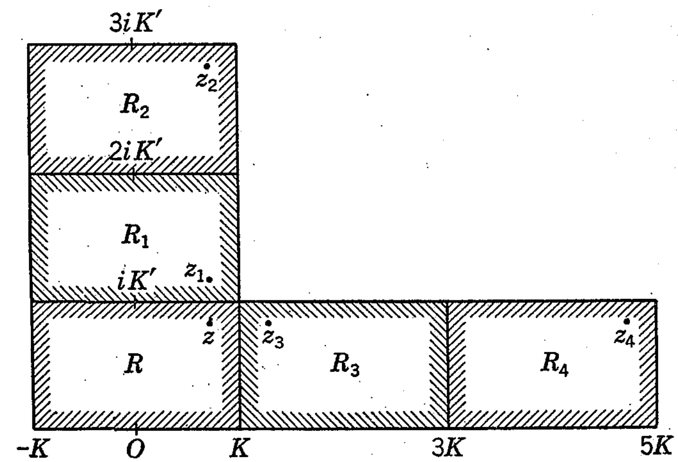

FIG. 35.

In view of the double periodicity (22), it is sufficient to know the values which the function $w = \text{sn}\, z$ takes in a rectangle of sides $4K$ and $2K'$, which are parallel to the x-axis and the y-axis, respectively. This is entirely analogous to the case of the function $w = \sin z$ in which it is sufficient to know the values of the function in a strip of width 2π which is parallel to the y-axis; the other values of $w = \sin z$ are then obtained from the relation $\sin(z + 2\pi n) = \sin z$, $n = \pm1, \pm2, \ldots$. We add that, in order to cover the entire z-plane by the homologues of the original *period rectangle*, it is necessary to add to the interior of the rectangle two of its sides; clearly, the latter have to be adjacent to each other. In the case of the function $w = \text{sn}\, z$, a convenient period rectangle will be the rectangle $-K \le \text{Re}\,\{z\} < 3K$, $0 \le \text{Im}\,\{z\} < 2K'$.

The period rectangle of $w = \text{sn}\, z$ consists of four homologues of the rectangle R in Fig. 35. As shown before, two of these rectangles are each mapped onto the half-plane $\text{Im}\{w\} > 0$ and the other two are mapped onto the half-plane $\text{Im}\{w\} < 0$. It follows that the function $w = \text{sn } z$ maps the period rectangle onto a domain which covers the entire w-plane exactly twice. In other words, *if w_0 is an arbitrary complex number, then the equation* $\text{sn } z = w_0$ *has exactly two solutions in each period rectangle.* This is also true of the equation $\text{sn } z = \infty$. By (14), $\text{sn } iK' = \infty$. If we invert the rectangle R in Fig. 35 with respect to its right side, the point $z = iK'$ is carried into the point $z = 2K + iK'$. Since the corresponding inversion with respect to the real axis in the w-plane leaves the point $w = \infty$ unchanged, it follows that $\text{sn}(2K + iK') = \infty$. The singularities of $\text{sn } z$ at $z = iK'$ and $z = 2K + iK'$ must be simple poles. This follows either by observing that the mapping at these points is conformal, or else by showing that otherwise there would exist values near $w = \infty$ which are taken in the period rectangle more than twice. Hence, the only singularities of $w = \text{sn } z$ in our period rectangle are two simple poles at the points $z = iK'$ and $z = 2K + iK'$.

Since, in view of (22), the values of the function $w = \text{sn } z$ in the entire z-plane are periodic repetitions of its values in a single period rectangle, it follows that *the only finite singularities of $w = \text{sn } z$ are simple poles at the points*

(23) $$z = 2nK + (2m + 1)iK', \qquad n = 0, \pm 1, \pm 2, \ldots, \quad m = 0, \pm 1, \pm 2, \ldots.$$

The zeros of $w = \text{sn } z$ are found in a similar fashion. By (14), $\text{sn } 0 = 0$. One inversion of the rectangle R in Fig. 35 with respect to its right side shows that also $\text{sn } 2K = 0$. These are all the zeros in the period rectangle. It follows therefore from (22) that *the only zeros of $w = \text{sn } z$ are simple zeros at the points*

(24) $$z = 2nK + 2miK', \qquad n = 0, \pm 1, \pm 2, \ldots, \quad m = 0, \pm 1, \pm 2, \ldots.$$

We shall use (23) and (24) in order to set up an infinite product for the function $w = \text{sn } z$. However, before we do so, we insert a few remarks concerning infinite products. An infinite product

$$P = \prod_{n=1}^{\infty} (1 + a_n), \qquad a_n \neq -1,$$

is defined as the limit of the products

$$P_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n) = \prod_{\nu=1}^{n} (1 + a_\nu)$$

for $n \to \infty$. If this limit exists and is different from zero, we say that the infinite product converges. The product is said to converge absolutely, if the product

$$p = \prod_{n=1}^{\infty} (1 + |a_n|)$$

converges. An absolutely convergent product converges also in the usual sense. To show that this is true, write

$$p_n = \prod_{\nu=1}^{n} (1 + |a_\nu|)$$

and consider the expressions

$$P_n - P_{n-1} = (1 + a_1) \cdots (1 + a_{n-1})a_n$$

and

$$p_n - p_{n-1} = (1 + |a_1|) \cdots (1 + |a_{n-1}|)|a_n|.$$

We obviously have

$$|P_n - P_{n-1}| \leq p_n - p_{n-1}.$$

Since $\lim_{n\to\infty} p_n = p$, $\sum_n (p_n - p_{n-1})$ converges. Hence, the same is true of $\sum_n |P_n - P_{n-1}|$ and therefore also of $\sum_n (P_n - P_{n-1})$. But the convergence of the latter series is identical with the existence of $\lim_{n\to\infty} P_n$.

It remains to be shown that this limit cannot be zero. Before we do so, we first prove that the infinite product $\prod_n (1 + |a_n|)$ and the infinite series $\sum_n |a_n|$ converge and diverge together. Obviously,

$$e^{|a_n|} = 1 + |a_n| + \frac{|a_n|^2}{2!} + \cdots \geq 1 + |a_n|.$$

On the other hand, we have

$$|a_1| + \cdots + |a_n| \leq (1 + |a_1|) \cdots (1 + |a_n|),$$

as is seen by multiplying out the product. Hence,

$$\sum_{\nu=1}^{n} |a_\nu| \leq \prod_{\nu=1}^{n} (1 + |a_\nu|) \leq e^{\sum_{\nu=1}^{n} |a_\nu|},$$

which shows that for the convergence of $\prod_n (1 + |a_n|)$ it is necessary and sufficient that the series $\sum_n |a_n|$ converge. To show now that $\lim_{n\to\infty} P_n \neq 0$, we remark that since $\sum_n |a_n|$ converges and $1 + a_n \to 1$, the series

$$\sum_n \left|\frac{a_n}{1+a_n}\right|$$

is also convergent. Hence, in view of the result just proved, the product

$$\prod_{\nu=1}^{n}\left(1 - \frac{a_\nu}{1+a_\nu}\right) = \frac{1}{\prod_{\nu=1}^{n}(1+a_\nu)} = \frac{1}{P_n}$$

tends to a finite limit. Therefore, $\lim_{n\to\infty} P_n \neq 0$.

We now consider the infinite product

$$f(z) = \zeta\,\frac{\prod_{m=0}^{\infty}(1 - q^{2m}\zeta^{-2})\prod_{m=1}^{\infty}(1 - q^{2m}\zeta^{2})}{\prod_{m=0}^{\infty}(1 - q^{2m+1}\zeta^{-2})\prod_{m=0}^{\infty}(1 - q^{2m+1}\zeta^{2})}, \tag{25}$$

where

$$\zeta = e^{\frac{\pi i z}{2K}} \tag{26}$$

and q is defined by (16). The product (25) converges absolutely at all points of the z-plane at which none of the terms in the numerator and denominator of (25) vanishes. Indeed, it follows from (16) that $0 < q < 1$. Hence, the series $\sum_m q^m$ converges; in view of the criterion just developed, the four products in (25) will therefore converge absolutely. We add that, by taking logarithms, an infinite product of the type

$$Q = \prod_{m=1}^{\infty}(1 - q^m\zeta)$$

can be transformed into the series

$$\log Q = \sum_{m=1}^{\infty}\log(1 - q^m\zeta),$$

if the proper values of the logarithm are taken on both sides. The series will obviously converge if ultimately, that is, for large enough m, we take

the value of $\log(1 - q^m\zeta)$ which reduces to 0 if $(1 - q^m\zeta) \to 1$. Since in any closed domain in which $q^m\zeta \neq 1$, $m = 1, 2, \ldots$, the series is converging, it converges there uniformly. As shown in Sec. 3, Chap. III, it therefore represents there a regular analytic function of ζ. Taking exponentials, we find that the same is true of $Q = Q(\zeta)$. In view of (26), it therefore follows that (25) is an analytic function of z which is regular at all finite points of the z-plane at which none of the factors in the denominator vanish. Obviously, all finite singularities of the function (25) are poles which are caused by the zeros of the denominator.

We now determine the zeros and poles of the function (25). In view of (16) and (26), the zeros coincide with those points z at which either

$$e^{-\frac{2\pi mK'}{K} - \frac{\pi i z}{K}} = 1, \qquad m = 0, 1, 2, \ldots, \tag{27'}$$

or

$$e^{-\frac{2\pi mK'}{K} + \frac{\pi i z}{K}} = 1, \qquad m = 1, 2, \ldots. \tag{27''}$$

Taking logarithms and observing the indeterminacy of the logarithmic function, we find that the zeros of (25) coincide with the points z for which

$$-\frac{2\pi mK'}{K} - \frac{\pi i z}{K} = 2\pi i n, \qquad m = 0, 1, 2, \ldots, n = 0, \pm 1, \pm 2, \ldots,$$

or

$$-\frac{2\pi mK'}{K} + \frac{\pi i z}{K} = 2\pi i n, \qquad m = 1, 2, \ldots, n = 0, \pm 1, \pm 2, \ldots.$$

Obviously, the last two relations are equivalent to

$$z = 2nK + 2imK', \qquad n = 0, \pm 1, \pm 2, \ldots, m = 0, \pm 1, \pm 2, \ldots.$$

A comparison with (24) shows that these are precisely the zeros of sn z. We add that these zeros of the function (25) are all simple; this is an immediate consequence of the fact that the derivatives of the left-hand sides of (27) do not vanish for any finite value of z. The poles of (25) are situated at the points z for which

$$e^{-\frac{\pi(2m+1)K'}{K} - \frac{\pi i z}{K}} = 1, \qquad m = 0, 1, 2, \ldots,$$

or

$$e^{-\frac{\pi(2m+1)K'}{K} + \frac{\pi i z}{K}} = 1, \qquad m = 0, 1, 2, \ldots.$$

As before, we find that these are the points

$$z = 2nK + (2m+1)iK', \qquad n = 0, \pm 1, \pm 2, \ldots, \quad m = 0, \pm 1, \pm 2, \ldots.$$

Since all these are simple zeros of the denominator, it follows that the only finite singularities of the function $f(z)$ in (25) are simple poles whose location is the same as that of the simple poles (23) of sn z.

We next show that the function (25) is a doubly periodic function whose periods are $4K$ and $2iK'$. In view of

$$e^{\frac{\pi i}{2K}(z+4K)} = e^{\frac{\pi i z}{2K}} \cdot e^{2\pi i} = e^{\frac{\pi i z}{2K}},$$

the quantity ζ defined in (26) does not change if z is replaced by $z + 4K$. Hence, the function (25) has the period $4K$. If z is replaced by $z + 2iK'$, we have

$$e^{\frac{\pi i}{2K}(z+2iK')} = e^{\frac{\pi i z}{2K}} e^{-\pi\frac{K'}{K}} = q e^{\frac{\pi i z}{2K}},$$

where q is defined by (16). As a result, the quantity ζ in (25) has now to be replaced by $q\zeta$. We obtain

$$f(z + 2iK') = q\zeta \frac{\prod_{m=0}^{\infty} (1 - q^{2(m-1)}\zeta^{-2}) \prod_{m=1}^{\infty} (1 - q^{2(m+1)}\zeta^2)}{\prod_{m=0}^{\infty} (1 - q^{2m-1}\zeta^{-2}) \prod_{m=0}^{\infty} (1 - q^{2m+3}\zeta^2)}$$
$$= qf(z) \frac{(1 - q^{-2}\zeta^{-2})(1 - q\zeta^2)}{(1 - q^2\zeta^2)(1 - q^{-1}\zeta^{-2})} = f(z).$$

Consider now the function

$$g(z) = \frac{\text{sn } z}{f(z)},$$

where $f(z)$ is the function defined in (25). Since the poles and zeros of sn z coincide with the poles and zeros, respectively, of $f(z)$, the function $g(z)$ is regular at all finite points of the z-plane. In view of the fact that both sn z and $f(z)$ have the periods $4K$ and $2iK'$, it further follows that $g(z)$ likewise has these periods. The values of $g(z)$ throughout the z-plane are therefore repetitions of the values taken in a single period rectangle. Since $g(z)$ is regular in the closure of such a rectangle, we have there $|g(z)| < M$, where M is a suitable constant. Hence, the inequality $|g(z)| < M$ must hold in the entire z-plane. In view of Liouville's theorem (Sec. 7, Chap. III), this means that $g(z)$ reduces to a constant, say C.

We have thus shown that

(28) $$\text{sn } z = Cf(z),$$

where $f(z)$ is defined in (25). To determine the constant C, we use the

fact that, by (14),

$$\text{sn } K = 1, \qquad \text{sn}(K + iK') = k^{-1}.$$

In these cases, the quantity ζ defined in (26) takes the values

$$e^{\frac{\pi i}{2K}K} = e^{\frac{\pi i}{2}} = i$$

and

$$e^{\frac{\pi i}{2K}(K+iK')} = e^{\frac{\pi i}{2}} e^{-\pi\frac{K'}{2K}} = i\sqrt{q},$$

respectively. Inserting these values in (25) and using (28) we obtain, after some manipulation,

(29) $$1 = 2iC \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}}\right)^2$$

and

$$\frac{1}{k} = \frac{iC}{2\sqrt{q}} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n-1}}{1 + q^{2n}}\right)^2.$$

Eliminating C from these two identities, we have

(30) $$k^2 = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}}\right)^8,$$

whence $C^4k^2 = q$. Since $q > 0$, it follows from (29) that $iC > 0$. Hence, finally

(31) $$C = -i\frac{\sqrt[4]{q}}{\sqrt{k}},$$

where both radicals take their positive values. In view of (25), (28), and (31), we thus have proved the expansion

(32) $$\text{sn}(z;q) = -i\frac{\sqrt[4]{q}}{\sqrt{k}} \zeta \frac{\prod_{n=0}^{\infty} (1 - q^{2n}\zeta^{-2}) \prod_{n=1}^{\infty} (1 - q^{2n}\zeta^2)}{\prod_{n=0}^{\infty} (1 - q^{2n+1}\zeta^{-2}) \prod_{n=0}^{\infty} (1 - q^{2n+1}\zeta^2)},$$

where ζ is defined by (26) and k, the *modulus* of the function sn z, is expressed in terms of q by means of (30).

Although so far our considerations have been confined to the case in which one of the periods of the function sn z is real and the other is pure imaginary, our results are of much wider application. The infinite product (32) converges for all values of q for which $|q| < 1$. Hence, if ω_1 and ω_2 are two complex numbers such that

$$(33)\qquad \operatorname{Re}\left\{\frac{\omega_1}{\omega_2}\right\} > 0$$

and, in analogy to (16) and (26), q and ζ are defined by

$$(34)\qquad q = e^{-2\pi\frac{\omega_1}{\omega_2}}, \qquad \zeta = e^{\frac{\pi i z}{\omega_2}},$$

the product (32) will converge for this value of q. The reader will also confirm without difficulty that this function $\operatorname{sn} z = \operatorname{sn}(z;q)$ has the periods $2\omega_1$ and $2\omega_2$, that is, it satisfies the relations $\operatorname{sn}(z + 2\omega_1) = \operatorname{sn} z$, $\operatorname{sn}(z + 2\omega_2) = \operatorname{sn} z$. The period rectangle will now become a *period parallelogram* which, because of (33), cannot degenerate into a linear segment. For $|q| < 1$, the expression (32) is also a regular analytic function of the variable q. By the principle of permanence (Sec. 5, Chap. III) all analytic identities which were shown to hold in the case $0 < q < 1$ will therefore persist for all q such that $|q| < 1$. We thus have the following more general result.

The analytic function $\operatorname{sn}(z;q)$ *of* (32), *where* ω_1, ω_2 *satisfy* (33) *and* q *and* ζ *are defined in* (34), *has the periods* $2\omega_1$ *and* $2\omega_2$*; the inverse of* $w = \operatorname{sn}(z;q)$ *is the function* $z = F(w)$ *defined in* (13).

Since these general functions $\operatorname{sn}(z;q)$ do not share the simple conformal mapping properties which are characteristic of the case $0 < q < 1$, we shall not pursue their study any further.

Infinite product expansions for the functions $\operatorname{cn} z$ and $\operatorname{dn} z$ defined in (17) and (18), respectively, can be obtained by suitable modification of the procedure employed in the case of $\operatorname{sn} z$. In view of (17), the poles of $\operatorname{cn} z$ coincide with those of $\operatorname{sn} z$, while its zeros are situated at the points at which $\operatorname{sn} z = \pm 1$. By (14), these are the points

$$(35)\qquad z = \pm K + 4Kn + 2iK'm, \qquad n, m = 0, \pm 1, \pm 2, \ldots .$$

It should be noted that the equation $\operatorname{sn}^2 z - 1 = 0$ has double roots at these points, or, what amounts to the same thing, that the derivative of $\operatorname{sn} z$ vanishes there. This is an immediate consequence of the fact (see Fig. 34) that $w = \operatorname{sn} z$ transforms right angles whose vertices are at these points into the angle π. Hence, $\sqrt{1 - \operatorname{sn}^2 z}$ is regular at these points and $\operatorname{cn} z$ is single-valued for all z. Using the fact that $\operatorname{cn} z$ has simple zeros at the points (35) and simple poles at the points (23), and employing the same procedure as in the case of the function $\operatorname{sn} z$, we finally arrive, after some manipulation, at the infinite product expansion

$$(36)\qquad \operatorname{cn}(z;q) = \frac{\sqrt[4]{q(1-k^2)}}{\sqrt{k}}\,\zeta\,\frac{\prod_{n=0}^{\infty}(1 + q^{2n}\zeta^{-2})\prod_{n=1}^{\infty}(1 + q^{2n}\zeta^{2})}{\prod_{n=0}^{\infty}(1 - q^{2n+1}\zeta^{-2})\prod_{n=0}^{\infty}(1 - q^{2n+1}\zeta^{2})}.$$

The function $\operatorname{dn} z$ of (18) has again the same poles as $\operatorname{sn} z$, while its zeros coincide with the points at which $\operatorname{sn} z = \pm k^{-1}$. By (14), these are the points

$$z = (2n + 1)K + (2m + 1)iK'.$$

This leads to the product expansion

$$(37)\qquad \operatorname{dn}(z;q) = \sqrt[4]{1-k^2}\,\frac{\prod_{n=0}^{\infty}(1 + q^{2n+1}\zeta^{-2})\prod_{n=0}^{\infty}(1 + q^{2n+1}\zeta^{2})}{\prod_{n=0}^{\infty}(1 - q^{2n+1}\zeta^{-2})\prod_{n=0}^{\infty}(1 - q^{2n+1}\zeta^{2})}.$$

Details of the derivation of (36) and (37) are left as an exercise to the reader.

Our next objective is to derive a relation between the functions $\operatorname{sn}(z;q)$ and $\operatorname{sn}(z;q^2)$. Our point of departure is again the conformal mapping of Fig. 34, by means of which the function $\operatorname{sn}(z;q)$ was originally defined.

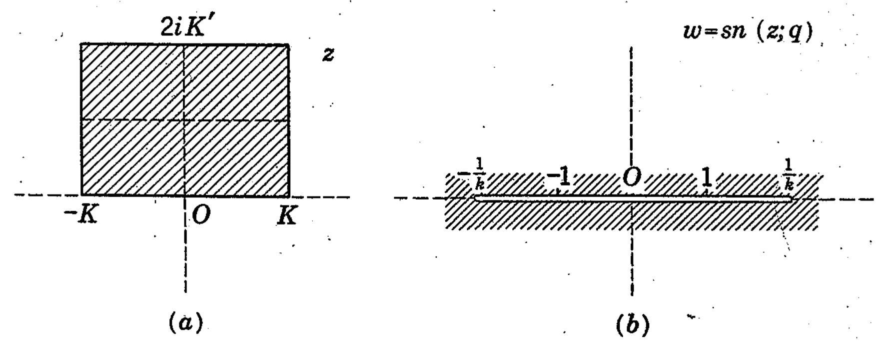

FIG. 36.

Applying the symmetry principle to an inversion of the rectangle in Fig. 34a with respect to its upper side, we find that the function $w = \operatorname{sn}(z;q)$ maps the rectangle of Fig. 36a onto the full w-plane which is furnished with a slit as indicated in Fig. 36b. Consider now the function $\operatorname{sn} \alpha z$ (where α is such that $\operatorname{sn} \alpha K = 1$) which maps the rectangle in Fig. 36a onto the upper half-plane. Since the sides of the rectangle are $2K$ and

$2K'$, respectively, the parameter q_1 belonging to this function is, by (16),

$$q_1 = e^{-\pi \frac{2K'}{K}} = q^2.$$

It follows that the function $w_1 = \text{sn}(\alpha z; q^2)$ maps the rectangle of Fig. 36*a* onto the upper half-plane. Since, as the reader will easily confirm, the mapping

$$w = \frac{2}{k} \frac{\beta w_1}{\beta^2 + w_1^2}, \qquad \beta > 0,$$

transforms the upper half-plane Im $\{w_1\} > 0$ into the slit domain of Fig. 36*b*, we find that the function

(38) $$w = \frac{2}{k} \frac{\beta\, \text{sn}(\alpha z; q^2)}{\beta^2 + \text{sn}^2(\alpha z; q^2)}$$

maps the rectangle of Fig. 36*a* onto the slit domain of Fig. 36*b*. As shown above, the same conformal mapping is effected by the function $w = \text{sn}(z;q)$. By the results of Sec. 4, Chap. V, two analytic functions which perform the same conformal mapping of a simply-connected domain are identical if their values agree in three boundary points. Now both the function (38) and $w = \text{sn}(z;q)$ transform $z = 0$ into $w = 0$. If the constant β in (38) is so chosen that

(39) $$\frac{2\beta}{1 + \beta^2} = k,$$

it follows further that the mapping (38) transforms the points $z = \pm K$ into the points

$$w = \frac{2}{k} \frac{\beta\, \text{sn}(\pm \alpha K; q^2)}{\beta^2 + \text{sn}^2(\pm \alpha K; q^2)} = \pm \frac{2}{k} \frac{\beta}{1 + \beta^2} = \pm 1.$$

Since also $\text{sn}(\pm K; q) = \pm 1$, the two functions must therefore be identical. We have thus proved the identity

(40) $$\text{sn}(z;q) = \frac{2}{k} \frac{\beta\, \text{sn}(\alpha z; q^2)}{\beta^2 + \text{sn}^2(\alpha z; q^2)},$$

where β is given by (39). The parameter k is the modulus of the elliptic function $\text{sn}(z;q)$. To distinguish k from the modulus of the function $\text{sn}(z;q^2)$, we shall denote the former by $k(q)$ and the latter by $k(q^2)$. In terms of this notation, the identity (30) reads

(41) $$k^2(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8.$$

The constant β in (40) can be simply expressed in terms of the modulus

$k(q^2)$. Since $\text{sn}[\alpha(K + 2iK'); q^2] = k^{-1}(q^2)$ and

$$\text{sn}(K + 2iK'; q) = \text{sn}(K;q) = 1,$$

it follows from (40) that

(42) $$k(q) = \frac{2\beta k(q^2)}{1 + \beta^2 k^2(q^2)}.$$

Now the equation

$$\frac{x}{1 + x^2} = \frac{y}{1 + y^2}$$

is identical with

$$(y - x)(1 - xy) = 0$$

and its only two solutions are therefore $x = y$ and $xy = 1$. Comparing (39) and (42), we thus find that we have either $\beta k(q^2) = \beta$ or $\beta^2 k(q^2) = 1$. Since the first possibility is absurd, it follows that

$$\beta = \frac{1}{\sqrt{k(q^2)}}.$$

Hence, in view of (39), (40), and the fact that $\text{sn}'(0) = 1$,

(43) $$k(q) = \frac{2\sqrt{k(q^2)}}{1 + k(q^2)},$$

(44) $$\text{sn}(z;q) = \frac{[1 + k(q^2)]\, \text{sn}(\alpha z; q^2)}{1 + k(q^2)\, \text{sn}^2(\alpha z; q^2)}, \qquad \alpha = [1 + k(q^2)]^{-1}.$$

By combining the mapping of Fig. 34 with some of the transformations discussed in the preceding sections, a number of important conformal

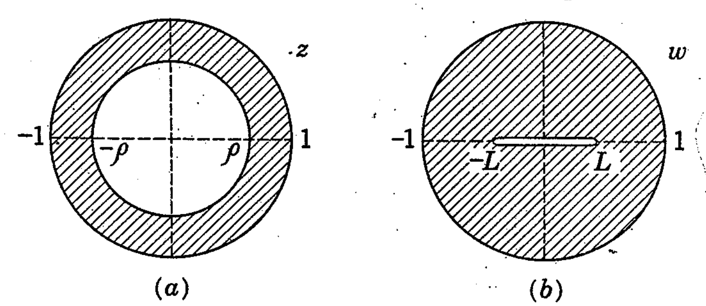

FIG. 37.

mappings can be obtained. As an example, we consider the conformal mapping of the circular ring of Fig. 37*a* onto the unit circle furnished with a symmetrical slit as indicated in Fig. 37*b*. It should be noted that the existence of such a mapping does not follow from the Riemann mapping theorem, since the domains are doubly-connected. It will be shown in

Chap. VII that any doubly-connected domain can be mapped onto a circular ring, where the ratio of the radii of the ring is completely determined by the original domain. In our case, this means that, given the length L in Fig. 37b, the radius ρ in Fig. 37a is completely determined, and vice versa.

Applying the symmetry principle to the circle $|z| = 1$, we find that the function $w = f(z)$ which effects the mapping of Fig. 37 will also map the circular ring $\rho < |z| < \rho^{-1}$ onto the entire w-plane which is furnished with the slits $-\infty \leq w \leq -L^{-1}$, $-L \leq w \leq L$, $L^{-1} \leq w \leq \infty$. Both these domains are symmetrical with respect to the real axis in their respective planes. If we can find a function which maps the upper half of the circular ring $\rho < |z| < \rho^{-1}$ onto the upper half-plane in such a way that the points

(45)

z	ρ	$-\rho$	ρ^{-1}	$-\rho^{-1}$
w	L	$-L$	L^{-1}	$-L^{-1}$

correspond to each other, it will therefore follow from the symmetry principle that this function is identical with $w = f(z)$. We now use the fact that the transformation $z = i\rho e^{-i\zeta}$ maps the rectangle $-\frac{1}{2}\pi < \text{Re}\,\{\zeta\} < \frac{1}{2}\pi$, $0 < \text{Im}\,\{\zeta\} < -2\log\rho$ onto the upper half of the circular ring $\rho < |z| < \rho^{-1}$ in such a way that the points

(46)

ζ	$\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi - 2i\log\rho$	$-\frac{1}{2}\pi - 2i\log\rho$
z	ρ	$-\rho$	ρ^{-1}	$-\rho^{-1}$

correspond to each other (see Fig. 32, Sec. 2). Hence, the function $w = f(i\rho e^{-i\zeta})$ maps the rectangle in question onto the upper half-plane. Since, in view of (45) and (46), we have the correspondence

ζ	$\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi - 2i\log\rho$	$-\frac{1}{2}\pi - 2i\log\rho$
$L^{-1}f(i\rho e^{-i\zeta})$	1	-1	L^{-2}	$-L^{-2}$

it follows from (14) that

$$L^{-1}f(i\rho e^{-i\zeta}) = \text{sn}\,\frac{2K}{\pi}\,\zeta,$$

where the constants K and K' associated with the function sn ζ satisfy

$$\frac{K'}{K} = -\frac{4}{\pi}\log\rho, \tag{47}$$

and where the modulus of the function sn is

$$k = L^2. \tag{48}$$

Since, in view of (16) and (47), the parameter q associated with sn $2\pi^{-1}K\zeta$ is ρ^4, we thus find that the analytic function effecting the conformal mapping of Fig. 37 is of the form

$$w = f(z) = \sqrt{k(\rho^4)}\,\text{sn}\left(\frac{2iK}{\pi}\log\frac{z}{\rho} + K;\,\rho^4\right). \tag{49}$$

We also note that, by (41) and (48), the functional dependence of the length L in Fig. 37b with respect to the radius is given by

$$L(\rho) = 2\rho\prod_{n=1}^{\infty}\left(\frac{1+\rho^{8n}}{1+\rho^{8n-4}}\right)^2.$$

As another example of a conformal mapping which can be carried out by means of elliptic functions, we consider the function $w = f(z)$ which maps the interior of an ellipse onto the unit circle. In order to avoid unnecessary parameters, we assume that the foci of the ellipse are situated at the points ± 1. If we further suppose—as we may, by the Riemann mapping theorem—that $f(0) = 0$, $f'(0) > 0$ then, for reasons of symmetry, $w = f(z)$ will map the upper half of the ellipse onto the upper half of the unit circle. We now recall from Sec. 2 that the function $z = \sin\zeta$ maps the rectangle $-\frac{1}{2}\pi < \text{Re}\,\{\zeta\} < \frac{1}{2}\pi$, $0 < \text{Im}\,\{\zeta\} < c$ onto the upper half of an ellipse of semiaxes $\cosh c$ and $\sinh c$. If the semiaxes of our ellipse are $a = \cosh c$, $b = \sinh c$, it follows therefore that $w = f(\sin\zeta)$ maps this rectangle onto the upper half of the unit circle $|w| < 1$. But this half circle is transformed by the mapping

$$w_1 = \frac{2w}{1+w^2}$$

onto the upper half-plane Im $\{w_1\} > 0$. Hence, the function

$$w_1 = \frac{2f(\sin\zeta)}{1+f^2(\sin\zeta)}$$

will map the rectangle $-\frac{1}{2}\pi < \text{Re}\,\{\zeta\} < \frac{1}{2}\pi$, $0 < \text{Im}\,\{\zeta\} < c$ onto the upper half-plane. If $f(1) = \alpha$, the reader will confirm that the corners of the rectangle correspond to the points

ζ	$\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi + ic$	$-\frac{1}{2}\pi + ic$
w_1	$\frac{2\alpha}{1+\alpha^2}$	$-\frac{2\alpha}{1+\alpha^2}$	1	-1

In view of (14), it follows therefore that

$$w_1 = \frac{2f(\sin\zeta)}{1+f^2(\sin\zeta)} = \frac{2\alpha}{1+\alpha^2}\,\text{sn}\,\frac{2K}{\pi}\,\zeta, \tag{50}$$

where the constants K and K' associated with the function sn satisfy $\pi K' = 2cK$. By (16), we have $q = e^{-2c}$, and the modulus $k = k(e^{-2c})$ is

$$k(e^{-2c}) = \frac{2\alpha}{1+\alpha^2}.$$

In view of (43), it follows that

$$\alpha = \sqrt{k(e^{-4c})}.$$

If we compare (50) with (44) and use (43), we find that

$$f(\sin\zeta) = \sqrt{k(e^{-4c})}\,\mathrm{sn}\left(\frac{2K}{\pi}\zeta;\, e^{-4c}\right), \qquad K = K(e^{-4c}).$$

If we recall that the semiaxes of the ellipse were $a = \cosh c$ and $b = \sinh c$, we thus obtain the following result.

The interior of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a^2 - b^2 = 1, \qquad z = x + iy$$

is mapped onto the unit circle $|w| < 1$ *by the function*

$$(51) \qquad w = f(z) = \sqrt{k(\rho)}\,\mathrm{sn}\left(\frac{2K}{\pi}\sin^{-1} z;\, \rho\right), \qquad \rho = \left(\frac{a-b}{a+b}\right)^2.$$

In this mapping, the foci of the ellipse correspond to the points

$$w = \pm\sqrt{k(\rho)}.$$

EXERCISES

1. Show that the elliptic functions sn z, cn z, dn z have the differentiation formulas

$$\frac{d}{dz}\,\mathrm{sn}\, z = \mathrm{cn}\, z\, \mathrm{dn}\, z,$$
$$\frac{d}{dz}\,\mathrm{cn}\, z = -\mathrm{sn}\, z\, \mathrm{dn}\, z,$$
$$\frac{d}{dz}\,\mathrm{dn}\, z = -k^2\,\mathrm{sn}\, z\, \mathrm{cn}\, z.$$

2. Show that $w = \mathrm{sn}\, z$ satisfies the differential equation

$$w'^2 = (1 - w^2)(1 - k^2w^2),$$

and deduce similar differential equations for the functions $w = \mathrm{cn}\, z$ and $w = \mathrm{dn}\, z$.

3. With the help of (41), (37), and the fact that dn $0 = 1$, prove the identity

$$\prod_{n=1}^{\infty}(1 + q^{2n-1})^8 - \prod_{n=1}^{\infty}(1 - q^{2n-1})^8 = 16q\prod_{n=1}^{\infty}(1 + q^{2n})^8, \qquad |q| < 1.$$

4. Show that the function $w = \mathrm{cn}\, z$ maps the rectangle $-K < \mathrm{Re}\,\{z\} < K$, $0 < \mathrm{Im}\,\{z\} < K'$ onto the right half-plane $\mathrm{Re}\,\{w\} > 0$ which has been cut along the linear segment $0 < w \leq 1$.

5. Show that the function

$$w = \sqrt{\frac{1 - \mathrm{cn}\, z}{1 + \mathrm{cn}\, z}}$$

maps the rectangle $-K < \mathrm{Re}\,\{z\} < K$, $-K' < \mathrm{Im}\,\{z\} < K'$ onto the unit circle $|w| < 1$.

6. Show that the function

$$w = \mathrm{sn}\left(\frac{4K}{\pi}\tan^{-1} z\right)$$

maps the unit circle $|z| < 1$ onto a domain D with the following properties: D covers the entire w-plane (including $w = \infty$) an infinity of times, with the exception of the linear segments $-k^{-1} \leq w \leq -1$ and $1 \leq w \leq k^{-1}$ which are not covered at all; D has no branch points and it has no boundary points other than the points of the above linear segments.

7. If $w = f(z,\rho)$ denotes the function effecting the conformal mapping of Fig. 37, use the fact—following from the symmetry principle—that $w = f(z,\rho)$ maps the circular ring $\rho < |z| < \rho^{-1}$ onto the full w-plane with the slits $-\infty \leq w \leq -L^{-1}$, $-L \leq w \leq L$, $L^{-1} \leq w \leq \infty$, in order to deduce the relations

$$f(z\rho^{-1},\rho) = \frac{2}{L(\rho)}\frac{f(z,\rho^2)}{1 + f^2(z,\rho^2)}$$

and

$$L^2(\rho) = \frac{2L(\rho^2)}{1 + L^2(\rho^2)}.$$

Using the form (49) of the mapping function $f(z,\rho)$, show that these two relations are identical with the identities (44) and (43) respectively.

8. Show that the function $w = f(z,\rho)$ of the preceding exercise maps the circular ring $\rho^2 < |z| < 1$ onto a domain which consists of two replicas of the unit circle $|w| < 1$ and has two simple branch points at $w = \pm L(\rho)$.

9. If $f(z,\rho)$ is the function of Exercise 7, show that the transformation

$$w = \frac{1 + f\left(\frac{1+z}{1-z}, \rho\right)}{1 - f\left(\frac{1+z}{1-z}, \rho\right)}$$

maps the z-plane with two equally large circular holes indicated in Fig. 38a onto the

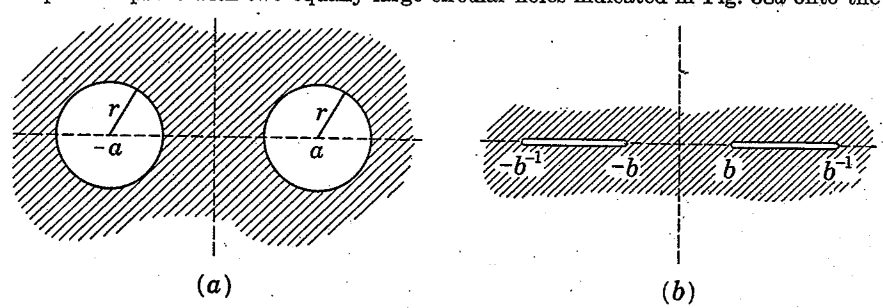

FIG. 38.

w-plane with two equal collinear slits indicated in Fig. 38b, where

$$a = \frac{1+\rho^2}{1-\rho^2}, \qquad r = \frac{2\rho}{1-\rho^2}, \qquad b = \frac{1-L(\rho)}{1+L(\rho)}.$$

10. Using the fact that the function (13) yields the conformal mapping of Fig. 34, prove the identity

$$\int_0^\pi \int_0^\infty \frac{\rho\, d\rho\, d\theta}{\sqrt{1-2\rho^2\cos 2\theta+\rho^4}\,\sqrt{1-2\rho^2k^2\cos 2\theta+\rho^4k^4}} = 2KK'.$$

11. If $K = K'$, it follows from (19), (20), and (21) that $k = 2^{-\frac{1}{2}}$. Deduce that $k(e^{-\pi}) = 2^{-\frac{1}{2}}$ and show further, with the help of (43), that

$$k(e^{-\frac{1}{2}\pi}) = 2\sqrt[4]{2}\,(\sqrt{2}-1).$$

12. Show that the transformation

$$w = \operatorname{sn}^2 z$$

maps the rectangle $0 < \operatorname{Re}\{z\} < K$, $-K' < \operatorname{Im}\{z\} < K'$ onto the full w-plane with the slits $-\infty \le w \le 0$, $1 \le w \le \infty$.

13. Show that the series

$$\vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z$$

and

$$\vartheta_4(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz, \qquad |q| < 1,$$

converge uniformly in any closed finite domain and thus are entire functions of z. Show further that

$$\vartheta_1(z+\pi) = -\vartheta_1(z), \qquad \vartheta_4(z+\pi) = \vartheta_4(z),$$
$$\vartheta_1(z+\pi\tau) = -q^{-1}e^{-2iz}\vartheta_1(z),$$
$$\vartheta_4(z+\pi\tau) = -q^{-1}e^{-2iz}\vartheta_4(z),$$

($q = e^{\pi i\tau}$) and that therefore the function

$$P(z) = \frac{\vartheta_1(z)}{\vartheta_4(z)}$$

satisfies the relations $P(z+\pi\tau) = P(z)$, $P(z+\pi) = -P(z)$ and thus is a doubly periodic function of z with the periods 2π and $\pi\tau$. *Hint:* Express the trigonometric functions in the definitions of ϑ_1 and ϑ_4 by means of exponential functions and replace the summations from 0 to ∞ by summations from $-\infty$ to ∞.

14. If C is the boundary of the parallelogram with the corners z_0, $z_0+\pi$, $z_0+\pi+\pi\tau$, $z_0+\pi\tau$ and $\vartheta(z)$ denotes either of the functions $\vartheta_1(z)$, $\vartheta_4(z)$, show that

$$\frac{1}{2\pi i}\int_C \frac{\vartheta'(z)}{\vartheta(z)}\,dz = \frac{1}{2\pi i}\int_{z_0}^{z_0+\pi}\left[\frac{\vartheta'(z)}{\vartheta(z)} - \frac{\vartheta'(z+\pi\tau)}{(z+\pi\tau)}\right]dz$$
$$-\frac{1}{2\pi i}\int_{z_0}^{z_0+\pi\tau}\left[\frac{\vartheta'(z)}{\vartheta(z)} - \frac{\vartheta'(z+\pi)}{\vartheta(z+\pi)}\right]dz = \frac{1}{2\pi i}\int_{z_0}^{z_0+\pi} 2i\,dz = 1.$$

Hint: Use the "quasi periodicity" of the function $\vartheta(z)$ derived in the preceding exercise.

15. Show that

$$\vartheta_1(z,q) = -ie^{iz+\frac{\pi i\tau}{4}}\vartheta_4\left(z+\frac{\pi\tau}{2}, q\right),$$

and deduce from the result of the preceding exercise that the only zeros and poles of the doubly periodic function $P(z)$ of Exercise 13 in a period parallelogram are situated at the points congruent to $z = 0, \pi$ and $z = \frac{1}{2}\pi\tau, \frac{1}{2}\pi\tau + \pi$, respectively.

16. Show that the zeros and poles of the function $P(z)$ coincide with those of $\operatorname{sn}(2K/\pi)z$ and that both functions have the same periods, and conclude that

$$P(z) = A\,\operatorname{sn}\frac{2K}{\pi}z,$$

where A is a constant.

17. Using the fact that $\operatorname{sn} K = 1$, $\operatorname{sn}(K+iK') = k^{-1}$, show that the value of the constant A in the preceding exercise is

$$A = \frac{2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}}{1+2\sum_{n=1}^{\infty} q^{n^2}}$$

and that

$$k^2 = 16q\left[\frac{\sum_{n=0}^{\infty} q^{n(n+1)}}{1+2\sum_{n=1}^{\infty} q^{n^2}}\right]^4.$$

Hint: Use the identity of Exercise 15. *Remark:* Because of the extremely rapid convergence of the series involved, this expression is used for the practical computation of k^2.

4. Domains Bounded by Arcs of Confocal Conics. In this section we consider the conformal mapping of domains whose boundaries consist of a number of confocal elliptic or hyperbolic arcs, where the common foci of these arcs may be assumed to be situated at the points ± 1 without restricting the generality of our considerations. Examples of such domains are the interior or exterior of an ellipse, the interior or exterior of one branch of a hyperbola, the entire plane slit along an elliptic or hyperbolic arc, a domain bounded by an elliptic arc and a hyperbolic arc intersecting it at right angles, and so forth.

Let now $w = f(z)$ be the analytic function which maps the unit circle $|z| < 1$ onto the domain D which is bounded by arcs of confocal conics whose common foci are at $w = \pm 1$. Since, as shown in Sec. 2, the mapping $\zeta = \sin^{-1} w$ transforms all ellipses and hyperbolas with the foci $w = \pm 1$ into linear segments parallel to the real and imaginary axes in

4. Let D be that part of an ellipse of foci ± 1 which is obtained by intersecting the ellipse with both branches of a hyperbola of foci ± 1, and which contains the origin. Show that the unit circle $|z| < 1$ is transformed into this domain D by the function

$$w = f(z) = \sin\left[C \int_0^z \frac{dz}{\sqrt{(z^2 - \alpha^2)(z^2 - \bar{\alpha}^2)}}\right],$$

where $|\alpha| = 1$ and α is neither real nor pure imaginary. Show also that this is equivalent to

$$z = \alpha \operatorname{sn}[C' \sin^{-1} w],$$

where $CC' = \bar{\alpha}$ and the modulus of the elliptic function sn is $k = \alpha^2$.

5. If, in the mappings of this section, the fundamental domain is taken to be the upper half-plane $\operatorname{Im}\{z\} > 0$ instead of the unit circle $|z| < 1$, show that (52) and (57) have to be replaced by the conditions

$$\frac{f'^2(z)}{1 - f^2(z)} = \text{real}, \qquad z = \text{real},$$

and

$$\frac{f'^2(z)}{f(z)} = \text{real}, \qquad z = \text{real},$$

respectively.

6. Let $w = f(z)$ be regular on a part L of the real axis, and let $f(z)$ satisfy there the condition

$$\frac{f'^4(z)}{[f(z) - a][f(z) - b][f(z) - c][f(z) - d]} \geq 0.$$

Show that $w = f(z)$ maps L onto an arc C with the following geometric property: If w is on C and β_1 and β_2 are the lines bisecting the angles between the lines connecting w with a, b and b, c, respectively, then the tangent to C at w makes equal angles with β_1 and β_2.

5. The Schwarzian s-functions. In Sec. 7, Chap. V, we discussed the analytic functions which map the upper half-plane onto a triangle whose sides are circular arcs. These functions are known by the name of the *Schwarzian s-functions* or the *Schwarzian triangle functions*. In the present section we shall be concerned with a further study of these functions, where the emphasis will be on the inverse to a given s-function rather than on the function itself. The reasons for doing so are similar to those for considering the elliptic function sn z rather than the elliptic integral (13) of which it is the inverse. While the integral (13) is an infinitely many-valued function, depending on the integration path connecting 0 and w, its inverse $w = \operatorname{sn} z$ is single-valued in its entire domain of existence, and this single-valuedness greatly facilitates all operations involving these functions. The s-functions are also infinitely many-valued. While it is not always true that the inverse of such a function is single-valued, there are important special cases in which the inverse has this property. It is to these cases that we shall devote our particular attention.

Taking account of the fact that we are now interested in the inverse of the s-function, we assume in this section that the curvilinear triangle with the angles $\pi\alpha$, $\pi\beta$, $\pi\gamma$ is situated in the z-plane. In this notation, the function $z = s(w)$ maps the upper half-plane $\operatorname{Im}\{w\} > 0$ onto the curvilinear triangle. The inverse to $z = s(w)$ will be denoted by $w = S(z)$ or, if it is desired to mention the angles of the triangle, by $w = S(\alpha,\beta,\gamma;z)$. In this definition, the function $w = S(z)$ is determined only up to an arbitrary linear transformation of the z-plane onto itself. To make things definite, we shall, by a suitable linear substitution, transform the triangle into such a position that the vertex with the angle $\pi\alpha$ is at the origin, while the two circular arcs meeting there become linear segments. Obviously, this is always possible if the angle $\pi\alpha$ is different from zero. Indeed, let C_1 and C_2 be the two circles which meet at $z = A$ under the angle $\pi\alpha \neq 0$. Since $\alpha \neq 0$, C_1 and C_2 intersect also at another point, say B. The linear substitution

$$z^* = \frac{z - A}{z - B}$$

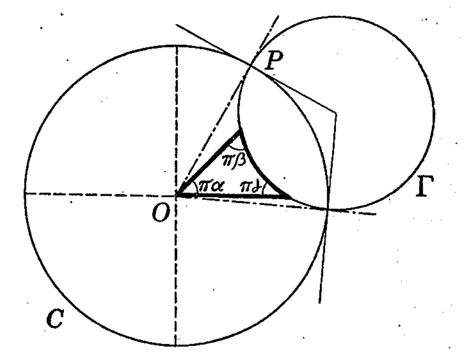

FIG. 40.

transforms all circles through B into straight lines; in particular, the circles C_1 and C_2 are transformed into two straight lines through the origin (both C_1 and C_2 pass through A). By an additional rotation, we may make one of these lines, say the line connecting $z = 0$ with the vertex at the angle $\pi\gamma$, coincide with the real axis.

In the following, we shall confine ourselves to triangles for which $\alpha + \beta + \gamma < 1$, that is, to triangles the sum of whose angles is less than 2π. Any such triangle has an *orthogonal circle*, that is, a circle which intersects the three circles making up the triangle at right angles. To show this, we bring the triangle into the position just mentioned and we observe that any circle about the origin is obviously orthogonal to the two rectilinear sides of the triangle. If Γ denotes the circle which forms the third side of the triangle, it follows from elementary considerations that the origin is in the exterior of Γ if $\alpha + \beta + \gamma < 1$. Hence, it is possible to draw a tangent from the origin to Γ (see Fig. 40). If P denotes the point of contact of this tangent, then the circle C about the origin which passes through P will obviously be the desired orthogonal circle. It is also clear that the triangle is entirely contained in the interior of the orthogonal circle. If one of the angles $\pi\beta$, $\pi\gamma$ is zero, then the corresponding vertex must be situated on the orthogonal circle. Indeed, the circle about the origin which passes through the vertex is orthogonal to one of the sides of the triangle meeting there; since the angle between the sides is 0, this circle is also orthogonal to the other side. If all three angles are zero, a case to be discussed in detail in the following section, all three vertices of the triangle are situated on the orthogonal circle.

We are mainly interested in those cases in which the functions

$$w = S(\alpha,\beta,\gamma;z)$$

are single-valued functions of the variable z. To find these cases, we observe that $w = S(z)$ maps two linear segments which meet at the origin under the angle $\pi\alpha$ onto a part of the real axis. As shown in detail in Sec. 6, Chap. V, this means that $S(z)$ is of the form

$$S(z) = z^{\frac{1}{\alpha}} S_1(z), \qquad S_1(0) \neq 0, \tag{58}$$

where $S_1(z)$ is regular at $z = 0$. But a power of z cannot be single-valued in the neighborhood of the origin unless its exponent is an integer. In view of $\alpha > 0$, this integer must be positive. $S(z)$ will therefore be single-valued in the neighborhood of $z = 0$ if, and only if, α is the reciprocal of a positive integer. The same result is obtained for β and γ by observing that, by suitable linear transformations, each of the other two vertices can be brought into the center of the orthogonal circle. Hence, a necessary condition for $w = S(z)$ to be single-valued is that α, β, γ be reciprocals of positive integers.

To show that this condition is also sufficient, we observe that, in view of the symmetry principle, all possible analytic continuations of $w = S(z)$ to points outside the original triangle can be obtained by successive inversions of the triangle in the z-plane and by corresponding inversions of the half-plane Im $\{w\} > 0$ with respect to the real axis. Now an inversion with respect to a circular arc transforms circles into circles; moreover, it preserves angles, although it reverses their orientation. Hence, any number of inversions of a circular triangle with respect to various circular arcs will again lead to a circular triangle with the same angles; if the number of inversions is even, the orientation of the angles will be the same as in the original triangle, while the orientation will be reversed if the number of inversions is odd. Since the corresponding inversions in the w-plane are simple symmetries with respect to the real axis, it follows that the boundaries of all these triangles are mapped onto parts of the real axis. By the argument employed in the case of the original triangle, we find therefore that $S(z)$ will be single-valued near the vertices of the inverted triangles if α, β, γ are the reciprocals of positive integers. On the other hand, these vertices are clearly the only possible singularities of $w = S(z)$. If α, β, γ are reciprocals of positive integers, it follows from (58) that $S(z)$ is regular at the vertices at which $S(z) = 0, 1$, while $S(z)$ has a pole of order $1/\beta$ at the vertices at which $S(z) = \infty$.

Summing up our results, we thus find that for all possible analytic continuations of $S(z)$, that is, in the entire domain of existence of this function, $S(z)$ has no singularities except poles of order β^{-1}. If we can show that the domain of existence of $S(z)$ is simply-connected, it will therefore follow from the monodromy theorem (Sec. 5, Chap. III) that $S(z)$ is indeed single-valued. (The fact that the function $S(z)$ has poles does not invalidate the reasoning leading to the proof of the monodromy theorem.) To find the domain of existence of $S(z)$ in the case $\alpha + \beta + \gamma < 1$, we depart from the observation made above that the fundamental triangle is situated within the orthogonal circle C. Since an inversion preserves the magnitudes of angles, it is clear that the sides of an inverted triangle are orthogonal to the image of C yielded by the inversion. But this image coincides with C, since an inversion with respect to a circular arc η transforms any circle orthogonal to η into itself. If we continue in this fashion, it is therefore clear that all the triangles which are obtained from the original triangle by successive inversions have the same orthogonal circle.

It follows that all these triangles are situated in the interior of the orthogonal circle C. This shows that $S(z)$ cannot be continued to points outside C. Within C, however, every point can be reached by a sufficient number of inversions. To see this, suppose that all possible successive inversions have been carried out. At the boundary of the domain covered by these triangles there can be no circular arcs of positive radius, since otherwise it would be possible to enlarge the domain by another inversion. It follows that the boundary of the domain is occupied by limit points of circular arcs whose radii tend to zero. But all these arcs are orthogonal to C, and therefore the circles to which they belong must intersect C. Since their radii tend to zero, the points of these arcs necessarily converge to C. This shows that any point within C can be reached by a sufficient number of inversions. Hence, *in the case $\alpha + \beta + \gamma < 1$, the domain of definition of $S(z)$ coincides with the interior of the orthogonal circle.* Since $S(z)$ cannot be continued beyond the circumference C of the orthogonal circle, C is a *natural boundary* of the function $S(z)$ as defined in Sec. 5, Chap. III.

As mentioned before, the function $S(z)$ has no singularities but poles if α, β, γ are the reciprocals of positive integers. Since, for $\alpha + \beta + \gamma < 1$, the domain of existence of $S(z)$ is a circle, *i.e.*, a simply-connected domain, it follows from the monodromy theorem that $S(z)$ is single-valued throughout its domain of existence. We thus have the following result.

If

$$\alpha = \frac{1}{m}, \quad \beta = \frac{1}{n}, \quad \gamma = \frac{1}{p}, \tag{59}$$

where

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{p} < 1, \tag{60}$$

and m, n, p are positive integers, then the function $w = S(\alpha,\beta,\gamma;z)$ is single-valued in the interior of the orthogonal circle C of the fundamental triangle. $S(z)$ cannot be continued beyond C.

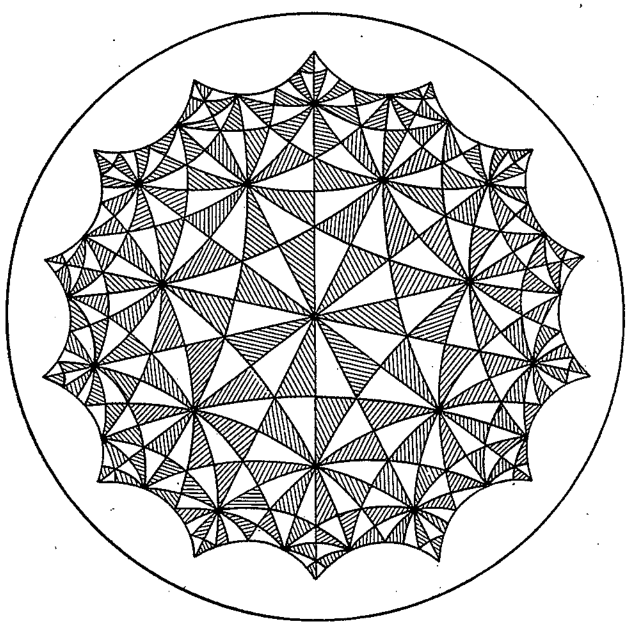

FIG. 41.

The geometric expression of the single-valuedness of the function $S(z)$ in these cases is the fact that the totality of the inversions of the fundamental triangle yields a simple covering of the interior of C. Figure 41 shows part of the triangular net obtained in the case $m = 2$, $n = 7$, $p = 3$, where the shaded and white triangles correspond to the upper and lower half-planes, respectively.

An inversion with respect to a circular arc preserves the magnitude of an angle but inverts its orientation. Hence, an even number of inversions preserves both the magnitude of an angle and its orientation, and it yields therefore a conformal mapping. Since, moreover, an inversion transforms circles into circles, it follows from the results of Sec. 2, Chap. V, that *an even number of inversions is equivalent to a linear transformation.* Since, in the w-plane, the corresponding inversions are simple symmetries with respect to the real axis, an even number of inversions in the w-plane will return us to our point of departure. If the linear transformation corresponding to an even number of inversions in the z-plane is of the form

$$z_1 = \frac{az + b}{cz + d},$$

it follows therefore that the function $w = S(z)$ satisfies the functional relation

$$S\left(\frac{az + b}{cz + d}\right) = S(z), \tag{61}$$

that is, the value of $w = S(z)$ is reproduced if the argument z is made subject to certain linear transformations. Functions with this property are known as *automorphic functions.* The various linear substitutions which reproduce the value of an automorphic function form a group. Indeed, if T and T' denote two such substitutions, it follows from $S(Tz) = S(z)$, $S(T'z) = S(z)$ that

$$S(TT'z) = S[T(T'z)] = S(T'z) = S(z).$$

Moreover, if T^{-1} is the substitution inverse to T, we have

$$S(z) = S(TT^{-1}z) = S(T^{-1}z).$$

The group belonging to a given function $S(z)$ can be constructed from three fundamental substitutions. Since all analytic continuations of $S(z)$ beyond the boundary of the fundamental triangle are obtained by inversions with respect to one of the sides of the triangle or by their combinations, it follows that any even number of inversions can be obtained by a suitable combination of the three linear transformations equivalent to the consecutive performance of two of the three fundamental inversions. The group can therefore be *generated* by three suitable substitutions (and, of course, the inverses of the latter). Every inversion with respect to one of the sides of a triangle leaves the orthogonal circle invariant, and the same is therefore true of the linear substitutions obtained by an even number of inversions. If we normalize the functions $S(z)$ by the requirement that the radius of the orthogonal circle be 1, and if we observe that the most general linear transformation of the unit circle onto itself is

$$z_1 = \delta\left(\frac{a - z}{1 - \bar{a}z}\right), \qquad |\delta| = 1, |a| < 1, \tag{62}$$

we arrive at the following result:

If the conditions (59) and (60) are satisfied, then $S(z)$ is a single-valued automorphic function in $|z| < 1$ which is invariant under a group of linear substitutions of the form (61), i.e., we have

$$f\left[\delta\left(\frac{z-a}{1-\bar{a}z}\right)\right] = f(z) \tag{63}$$

for the substitutions of this group. The group in question can be generated by three particular substitutions and their inverses.

We add a few remarks concerning the analytic expression of the function $w = S(z)$. It was shown in Sec. 7, Chap. V, that the inverse $z = s(w)$ of $w = S(z)$ is, up to an arbitrary linear transformation, of the form (75), Sec. 7, Chap. V. This expression does not, however, yield the inverse of the function $w = S(z)$ as normalized in the present section. To obtain the latter function—corresponding to the position of the fundamental triangle as indicated in Fig. 40—we have to go back to the fact, proved in Sec. 7, Chap. V, that the function mapping the upper half-plane onto a circular triangle with the angles $\pi\alpha$, $\pi\beta$, $\pi\gamma$ can be represented as the quotient of two linearly independent solutions of the hypergeometric differential equation

$$w(1-w)\zeta'' + [c - (1+a+b)w]\zeta' - ab\zeta = 0, \qquad \zeta' = \frac{d\zeta}{dw} \tag{64}$$

where

$$\alpha = 1 - c, \qquad \beta = b - a, \qquad \gamma = c - a - b. \tag{65}$$

As also shown in Sec. 7, Chap. V, (64) is solved by the hypergeometric series

$$F(a,b,c;w) = 1 + \frac{ab}{1!c}w + \frac{a(a+1)b(b+1)}{2!c(c+1)}w^2 + \cdots, \tag{66}$$

which converges for $|w| < 1$. A suitable second solution is obtained by the remark that by substituting

$$\zeta = w^{1-c}\zeta_1$$

in (64) we arrive at a differential equation for ζ_1 which differs from that for ζ only by the fact that a, b, c are now replaced by $a - c + 1$, $b - c + 1$, $2 - c$, respectively. Since (64) is solved by the function (66), it follows that another solution of (64) is given by

$$w^{1-c}F(a-c+1, b-c+1, 2-c; w), \qquad 0 < c < 1. \tag{67}$$

Since, for $w = 0$, this solution reduces to 0, while the solution (66) reduces to 1, these two solutions are linearly independent.

Consider now the quotient of the two solutions (66) and (67), i.e., the function

$$z = s(w) = \frac{w^{1-c}F(a-c+1, b-c+1, 2-c; w)}{F(a,b,c;w)}, \tag{68}$$

where a, b, c are connected with α, β, γ by (65), or, if we solve (65) for a, b, c, by

$$a = \tfrac{1}{2}(1 - \alpha - \beta - \gamma), \qquad b = \tfrac{1}{2}(1 - \alpha + \beta - \gamma), \qquad c = 1 - \alpha. \tag{69}$$

Since $c < 1$, it follows from (68) that $s(0) = 0$, which shows that one vertex of the circular triangle is situated at $z = 0$. The hypergeometric function (66) is obviously real if all its arguments are real. If we choose that determination of w^{1-c} which is real for positive w, it follows therefore from (68) that $s(w)$ is real if w varies along the real axis from $w = 0$ to $w = 1$. Hence, one side of the circular triangle is a part of the positive axis terminating at the origin. A second side of the circular triangle is obtained as the conformal image of the negative axis $-\infty < w < 0$. While the two hypergeometric functions in (68) are also real for these values of w, the factor w^{1-c} is now equal to

$$\begin{aligned} w^{1-c} &= (-|w|)^{1-c} = |w|^{1-c}(e^{\pi i})^{1-c} = |w|^{1-c}e^{\pi i(1-c)} \\ &= |w|^{1-c}e^{\pi i\alpha}. \end{aligned}$$

This shows that $-\infty < w < 0$ is mapped by $z = s(w)$ onto a linear segment which makes the angle $\pi\alpha$ with the real axis at the origin. We have thus proved that the circular triangle yielded by the function (68) has indeed the position indicated in Fig. 40.

We finally compute the coordinates of the vertices of the circular triangle upon which $\text{Im}\{w\} > 0$ is mapped by (68), that is, the values of $s(0)$, $s(1)$, $s(\infty)$. To find $w(1)$, we observe that, by (73), Sec. 7, Chap V,

$$\begin{aligned} F(a,b,c;1) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \end{aligned}$$

The condition $c - a - b > 0$, which is required for the existence of the integral, is satisfied since, by (65), $\gamma = c - a - b$. Using (68) we thus find

$$s(1) = \frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}. \tag{70}$$

This is the vertex with the angle $\pi\gamma$. Since $s(0) = 0$, it thus remains to find the vertex $s(\infty)$, corresponding to the angle $\pi\beta$. For this purpose we