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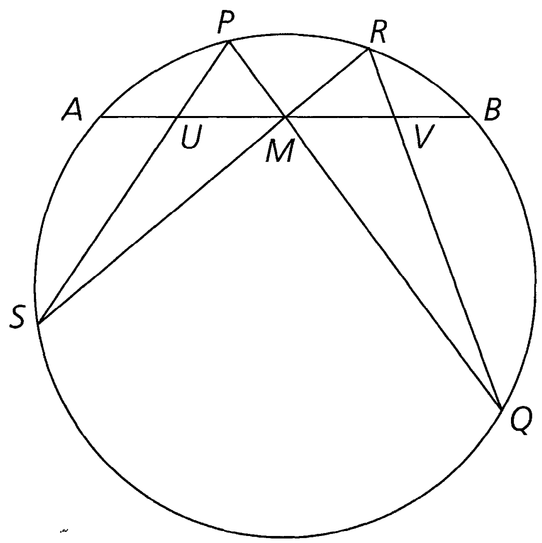
Mathematics and Ideologies

Now that we are able to distinguish between Euclidean, affine, and projective geometries, we may classify our knowledge accordingly. So, we have seen that Pappus's theorem belongs to projective geometry. But the Pythagorean theorem belongs to Euclidean geometry because it involves the concept of *length* of the sides of a triangle. Classification is a great source of satisfaction for scientists in general and mathematicians in particular. Classification is also useful: to understand a problem of Euclidean geometry, you will use one bag of tools including congruent triangles and the Pythagorean theorem. For a problem of projective geometry, you will use another bag of tools containing projective transformations and the fact that they preserve cross-ratios. A problem may be fairly easy if you use the right bag of tricks and become quite hard if you use the wrong one. Working mathematicians frequently experience this state of affairs and give due credit to Felix Klein for having uncovered this hidden mathematical reality: there are several different geometries, and it is useful to know where individual problems belong.

To convince you that the Erlangen program is a useful piece of mathematical ideology, I would like now to discuss a *difficult* problem. Here it is:



Butterfly Theorem
Draw a circle and a chord AB, with midpoint M. Then draw chords PQ and RS through M as in the picture. Finally, let the chords PS and RQ intersect AB at U and V, respectively. Claim: M is the midpoint of the segment UV.

(Note that the *butterfly PSRQ* is usually an asymmetric quadrangle.) If you have some training in elementary geometry, I recommend that you give this problem a good try before going on (stop reading, take a sheet of paper, and start working).

Now let me explain that a professional mathematician would not call this a very hard problem. With respect to difficulty, it has nothing to do with *Fermat's last theorem*, which we shall discuss later. In fact, it looks like an easy question of elementary Euclidean geometry. One immediately notices that the angles at *S* and *Q* are equal. Then one tries to use standard results about congruent triangles (like the one in chapter 2). Perhaps one makes some constructions, drawing a perpendicular or a bisector, . . . and one gets absolutely nowhere. Then one may start having doubts. Is it really true that *M* is the middle of *UV*? (In fact, yes, it is true.) The reasonable thing to do in this sort of situation is to sleep on it. (I am a most reasonable person, so that is exactly what I did after my colleague Ilan Vardi showed me the problem and I could not readily solve it.) If you really want to crack the problem, you can now do either of two things.

(i) Use brute force. In fact, for problems of elementary geometry one can always (as we shall see) introduce coordinates, write equations for the lines that occur, and reduce the problem to checking some algebra. This method is due to Descartes.[1] It is effective but cumbersome. It is often long and inelegant, and some mathematicians will say that it teaches you nothing: you don't get a real understanding of the problem you have solved.

(ii) Find a clever idea so that the problem becomes easy. To most mathematicians this is the preferred method.

Here the clever idea is to realize that the butterfly theorem belongs to projective rather than Euclidean geometry. True, the circle is a Euclidean object, but it also appears naturally in the geometry of the complex projective line. True, the notion of a midpoint is Euclidean or affine, but this is a red herring: one might start with $|AM| = \alpha |AB|$, where α is not necessarily $1/2$.

Let me now briefly outline a proof of the butterfly theorem. You may work out the details or be satisfied with the general idea, as you prefer. Consider the points $A, B, P, R, \ldots$ to be complex numbers. Since A, B, P, R are on a circle, the cross-ratio $(A, B; P, R)$ is real. Taking the origin of the complex plane at S, we find that the points $A' = 1/A$, $B' = 1/B$, $P' = 1/P$, $R' = 1/R$ are on a straight line and

$$(A, B; P, R) = (A', B'; P', R') .$$

This is also* the cross-ratio of the lines $SA', SB'; SP', SR'$ or (by a reflection) of the lines $SA, SB; SP, SR$ or (intersecting by AB) of the points $A, B; U, M$. Thus,

$$(A, B; P, R) = (A, B; U, M) .$$

Replacing S by Q we find similarly

$$(A, B; P, R) = (A, B; M, V) .$$

We have shown that

$$(A, B; U, M) = (A, B; M, V),$$

that is,

$$\frac{U-A}{M-A} : \frac{U-B}{M-B} = \frac{M-A}{V-A} : \frac{M-B}{V-B} .$$

If M is the midpoint of AB, that is, $M = (A + B)/2$, then the above equation simplifies and we have

$$(U - A)(V - A) = (U - B)(V - B),$$

or, expanding and regrouping,

$$(B - A)(U + V) = B^2 - A^2,$$

or, dividing by $B - A$,

$$U + V = A + B.$$

This shows that the midpoint of the segment UV is

* At this point we pass briefly from one-dimensional complex projective geometry to two-dimensional real geometry. By the way, instead of one-dimensional complex projective geometry, some mathematicians will prefer to use two-dimensional conformal geometry, but it is nearly the same thing.

$$\frac{U+V}{2} = \frac{A+B}{2} = M,$$

as announced.

So, we find that a difficult problem of geometry has a natural and elegant proof once we understand that the problem naturally belongs to projective rather than Euclidean geometry. This example, with many others, shows that there are natural structures in mathematics. These natural structures need not be easy to see. They are like the pure ideas or forms that Plato had imagined. The mathematician thus has access to the elegant world of natural structures, just as, in Plato's view, the philosopher can reach the luminous world of pure ideas. For Plato, in fact, a philosopher must also be a geometer. Today's mathematicians are thus the rightful descendants of the philosopher-geometers of ancient Athens. They have access to the same world of pure forms, eternal and serene, and share its beauty with the Gods. This sort of view of mathematics has come to be called *mathematical Platonism*. And in some form or other it remains popular with many mathematicians. Among other things it puts them above the level of common mortals. Mathematical Platonism can not be accepted uncritically, however, and we shall later dwell at length on this problem.

But at this point a shocking question should be discussed: how did our butterfly theorem find itself included in a list of "anti-Semitic problems"? The setting of the story is the Soviet Union, and the time is the 1970s and 1980s. As you may know, the Soviet Union was brilliant in many fields of science, in particular mathematics and theoretical physics. Scientific excellence was rewarded, and following the meanderings of the party line played a less dominant role in science than in other areas of Soviet life. Scientists then were to some extent sheltered from the prejudices of the ruling caste. But eventually the Soviet authorities, via the party committees of the universities, took steps to change this situation. In particular they limited the admission of Jews and certain other national minorities to major universities (in particular, Moscow University). This policy was not open and official but was implemented by selectively failing undesirable candidates at entrance examinations. Some details are given in papers by Anatoly Vershik and Alexander Shen,[2] where they